SIAM J. NUMER. ANAL.
Vol. 36, No. 3, pp. 953–973

NUMERICAL ANALYSIS OF CONSERVATIVE AND ENTROPY SCHEMES FOR THE FOKKER–PLANCK–LANDAU EQUATION*

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Abstract. Conservatives and entropy schemes for the Fokker–Planck–Landau (FPL) equation are studied. We prove the existence of a unique positive and global in time solution for the homogeneous linear and nonlinear discretized (either in the velocity space or both in the velocity space and in time) FPL equation. The stability analysis of these schemes leads to sufficient conditions on the time-step that guarantee positivity and entropy decay.

Key words. kinetic models, Fokker–Planck–Landau equation, system of ordinary differential equations, Cauchy problem, numerical schemes

AMS subject classifications. 34A10, 65L05, 65M05, 82A45

PII. S0036142997322102

1. Introduction.

The Fokker–Planck–Landau (FPL) equation describes the binary collisional effects (through long range Coulombian interaction) in a plasma [20] and can be derived from the Boltzmann equation in the grazing collision limit [11, 13, 1, 2]. Other applications may be found in astrophysics, for example, in the study of star cluster models [8, 9]. The linear FPL equation can also be used for diphasic flow modeling [10] and in space plasma physics for polar outflow modeling of a minor ion [23, 30]. Note that it is generally necessary to solve the nonlinear FPL equation to obtain quantitative agreement with experimental data [24]. The resulting prohibitive numerical cost forces many authors to consider simplified models, although rapid algorithms can now be used [7, 22].

In plasma physics, conservative schemes are of great importance since nonconservative designed schemes generally produce artificial heating or cooling of the plasma, as mentioned in [18] for the Boltzmann equation. Therefore, such methods require a great number of discretization points and thus, a prohibitive CPU time. We refer to [20] for such discretizations, which are based on the so-called Rosenbluth form of the FPL equation. In this paper, the major preoccupation is to reduce the cost of the algorithm using FFT method. In other words, entropy decay is important to ensure the thermalization of the plasma to the physically relevant temperature. Moreover, the methods considered here can be easily extended to multispecies plasma, as explained in [7], which is one of the major difficulties with the nonconservative schemes. Numerically, it can also be observed that entropy decay and positivity generally entail a nonoscillatory scheme. This property will be proved in the linear case. We refer to [6] for numerical evidence of these oscillations for the spherical Fokker–Planck equation when the entropy decay is not satisfied. Conservatives and entropy schemes for the nonlinear FPL equation can be only designed using the Landau and the so-called log formulation (which will be defined later on) of the equation, as proved by many

*Received by the editors May 27, 1997; accepted for publication (in revised form) April 13, 1998; published electronically May 5, 1999.

http://www.siam.org/journals/sinum/36-3/32210.html

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authors [12, 3, 27, 5, 25, 17, 26]. Indeed, we construct in the appendix a conservative scheme which is based on the Landau form but in the nonlog formulation for which a positive initial distribution function does not remain positive after an arbitrarily small time.

One open problem that is addressed here concerns existence of positive solutions for such discretizations. Another point of interest is the entropy decay of the fully discretized equation (i.e., also in time). Indeed, the entropy decay is guaranteed in the log Landau form only for the semidiscretized problem (i.e., discretized in the velocity space and continuous in time, which leads to a system of coupled nonlinear differential equations), but not for the fully discretized problem which is actually implemented in numerical simulations. Berezin, Khudick, and Pekker [3] provide an insight into these questions but their response remains incomplete; they give no rigorous proof either for the decay of the entropy for their time discretized scheme or for the existence of a positive (time-discretized or not) solution. In the present paper, we will prove some existence and stability results for the fully discretized and semidiscretized FPL equations in the homogeneous case, i.e., when there is no x-space dependence, using the Landau log formulation in both linear and nonlinear cases.

In section 2, we deal with the linear FPL equation, that is, a simpler situation in which it is easy to understand what happens when the Landau log formulation of the equation is discretized. We study the semidiscretized problem which leads to a system of ordinary differential equations (ODEs). These equations are nonlinear since the discretization is done on the log formulation in order to satisfy the entropy decay, as explained in the rest of this paper. We prove that this system still verifies a maximum principle and has a global in time solution with a uniform nonnegative lower bound. We show that the distribution function converges toward the discrete Maxwellian when $t \to \infty$. Then, we consider the time-discretized problem and we prove that the proposed explicit scheme has the same properties as the semidiscretized problem (and as the continuous one). More precisely, we construct a piecewise constant in time approximation of the velocity-discretized distribution function. This discretized solution remains positive, entropy decaying, nonoscillatory, and tends to the discrete Maxwellian for large time. These properties are achieved under some time-step restriction that provides a CFL-like condition for the scheme.

In section 3, we consider the three-dimensional nonlinear FPL equation, which is more interesting for plasma physics applications. We introduce a new conservative and entropy decaying discretization of the Fokker–Planck operator that guarantees the existence of a positive solution. More precisely, we prove that the semidiscretized problem has a global in time, positive, and entropy solution. However, we have no uniform, strictly positive lower bound for the solution in arbitrary large time and, for this reason, we cannot prove that the solution converges toward the velocity-discretized Maxwellian. Then, we consider the time-discretized problem. We prove that the sequence of time steps for which the solution remains positive at each iteration, forms a divergent series. Using this fact, we construct an approximate entropy solution for an arbitrarily large time. We also exhibit a lower bound for the time-step that ensures the decay of the entropy. Although this scheme is entropy decaying, the convergence of the discretized distribution function toward the discrete Maxwellian (or equivalently a lower-bound uniform in time on its weights) remains an open problem.

Finally, we point out that these first existence results of an entropy positive discretized solution can also be viewed as a first step toward proving any convergence or consistency result for these methods.

2. The linear FPL operator.

2.1. The continuous homogeneous linear FPL equation. Let us consider the linear FPL equation in dimension d, which is a second-order differential equation. This convection-diffusion equation can equivalently be written in the following forms (for strictly positive distribution functions):

$$\frac{\partial f}{\partial t} = \nabla \cdot \left((\vec{v} - \vec{u}) f + T \vec{\nabla}_v f \right) \tag{2.1}$$

$$= \nabla \cdot \left(T f \vec{\nabla}_v \log(f/M_f) \right) \tag{2.2}$$

$$= \nabla \cdot \left(T f \left(\frac{(\vec{v} - \vec{u})}{T} + \vec{\nabla}_v \log f \right) \right) \;=\; \nabla \cdot \left(T M_f \vec{\nabla}_v (f/M_f) \right),$$

where $\nabla\cdot$ and $\vec{\nabla}_v$ are the divergence and gradient operator, respectively, $f(v,t)$ is the distribution function (with $\vec{v} \in \mathbb{R}^d$, $t > 0$). T represents a constant temperature, $\vec{u}$ a constant velocity, and M_f the Maxwellian with the same first moment n_f as f, defined by

$$M_f(\vec{v}) = \frac{n_f}{(2\pi T)^{d/2}} \exp\left(\frac{-\|\vec{v} - \vec{u}\|^2}{2T} \right), \tag{2.3}$$

$$n_f(t) = \int_{\mathbb{R}^d} f(v,t) dv \quad \text{(mass)},$$

$$\vec{u}_f(t) = \frac{1}{n_f(t)} \int_{\mathbb{R}^d} \vec{v} f(v,t) dv \quad \text{(velocity)},$$

$$T_f(t) = \frac{1}{d\, n_f(t)} \int_{\mathbb{R}^d} \|\vec{v} - \vec{u}_f(t)\|^2 f(v,t) dv \quad \text{(temperature)}.$$

Equation (2.2) is called Landau-log formulation of the linear FPL equation. In the following, we assume $T = 1$ for the sake of simplicity. For any test-function $\phi(v)$, we have, integrating (2.2) by part

$$\int_{\mathbb{R}^d} \frac{\partial f}{\partial t} \phi(v) dv = - \int_{\mathbb{R}^d} f \vec{\nabla}_v \phi^T \cdot \vec{\nabla}_v \log(f/M_f) dv, \tag{2.4}$$

where x^T denotes the transpose vector. By taking $\phi \equiv 1$, we obtain the mass conservation which is the only conservation for this equation, i.e., $n_f(t) = n_f(0) = n$. Thus, the Maxwellian is constant in time. Note that, in general, this equation does not preserve momentum and energy. Indeed, letting $\phi = v$ in (2.4), we get

$$\frac{d}{dt} \left(\vec{u}_f(t) - \vec{u} \right) = -(\vec{u}_f(t) - \vec{u}),$$

where u is the constant velocity. Thus, we have

$$\vec{u}_f(t) = \vec{u}_f(0) \exp(-t) + \vec{u} \left(1 - \exp(-t) \right).$$

Therefore, the velocity is not constant except if $\vec{u}_f(0) = \vec{u}$. Let us now choose

$$\phi(\vec{v}) = \frac{1}{dn} \|\vec{v} - \vec{u}_f(t)\|^2,$$

in (2.4) or equivalently in the weak form of (2.1), we have ($T = 1$)

$$\frac{d}{dt}\left(T_f(t) - 1\right) = \frac{-1}{dn}\int_{\mathbb{R}^d} \vec{\nabla}_v \left(\|\vec{v} - \vec{u}_f(t)\|^2\right)^T \cdot \left((\vec{v} - \vec{u}_f)f + \vec{\nabla}_v f\right) dv$$
$$= 2 - \frac{2}{dn}\int f(\vec{v} - \vec{u}_f(t))^T \cdot (\vec{v} - \vec{u})dv = 2(1 - T_f(t)),$$

since $\vec{\nabla}_v\left(\|\vec{v} - \vec{u}_f(t)\|^2\right) = 2(\vec{v} - \vec{u}_f(t))$ and $\int f(\vec{v} - \vec{u}_f(t))dv = \vec{0}$. Thus, we have

$$T_f(t) = T_f(0)\exp(-2t) + (1 - \exp(-2t)),$$

i.e., the temperature tends to the equilibrium temperature $T = 1$. Using $\phi = \log(f/M_f)$, we obtain the H-theorem

$$H = \int_{\mathbb{R}^d} f\log(f/M_f)dv, \quad \frac{dH}{dt} = -\int_{\mathbb{R}^d} f\left\|\vec{\nabla}_v \log(f/M_f)\right\|^2 dv \le 0$$

and $dH/dt = 0 \Leftrightarrow f = M_f$. The existence of a solution for the Cauchy problem associated with (2.1) is classical. It can be easily shown that the solution remains strictly positive for nonnegative initial data. The remainder of this section is devoted to the existence proof of solution for the discretized problem using the log formulation, i.e., (2.2). We obtain a discrete analogous of the H-theorem and prove that the solution tends to the discretized Maxwellian for large time.

2.2. The semidiscretized linear FPL equation. We shall consider the problem discretized in the velocity space. Let us first discuss the discretization of the gradient, i.e., the choice of the finite difference operators.

2.2.1. Finite difference operator. Let D be a finite difference operator that approximates the usual gradient operator $\vec{\nabla}_v$ at least up to the first order defined as follows: for any test sequence $\psi = (\psi_i)_{i\in\mathbb{Z}^d}$, $D\psi$ is a sequence $(D\psi)_i \in \mathbb{Z}^d$ of vectors of $\mathbb{R}^d$ defined by

$$(D\psi)_i = ((D^s\psi)_i)_{s=1,\ldots,d} \in \mathbb{R}^d,$$

where the components $(D^s\psi)_i$ for $s = 1,\ldots,d$ approximate the partial derivatives $\partial\psi/\partial x_s(v_i)$. The s-component of such finite difference operator is of the form

$$(D^s\psi)_i = \sum_{k\in N^s} \alpha_k^s \psi_{i+k}, \tag{2.5}$$

where N^s is a finite set of indices $k \in \mathbb{Z}^d$ which contains the neighbors of the points involved for the s-component by the finite difference operator. We also set throughout the rest of the paper $N = \cup_{s=1,\ldots,d} N^s$. The vectors $\alpha_k = (\alpha_k^s)_{s=1,\ldots,d} \in \mathbb{R}^d$ satisfy the following symmetry properties:

$$\sum_{k\in N^s} \alpha_k^s = 0, \quad \sum_{k\in N^s} \alpha_k^s k^r \Delta v = \delta_{sr}, \quad s = 1,\ldots,d, \quad r = 1,\ldots,d, \tag{2.6}$$

where k^r is the rth component of the vector $k \in \mathbb{Z}^d$ and δ_{sr} is the Kronecker symbol. Condition (2.6) states that D is the exact gradient for constant and linear functions or equivalently, that D is an approximation of $\vec{\nabla}_v$ at least up to the first order. The formal adjoint D^* of D is given, for each component, by

$(D^{*s}\psi)_i = \sum_{-k\in N^s}\alpha_k^{*s}\psi_{i+k}$, with $\alpha_k^{*s} = \alpha_{-k}^s \quad \forall(-k) \in N^s$. There are two simple cases of such discretized operator. On the one hand, the 2^d uncentered difference operators denoted by D_ε, for $\varepsilon = (\varepsilon_j)_{j=1,\dots,d} \in \{-1,1\}^d$ defined by

$$(D_\varepsilon^s\psi)_i = \frac{1}{\Delta v}\left(\varepsilon_s(\psi_{i+\varepsilon_s e_s} - \psi_i)\right),$$

for such uncentered, we have $(D_\varepsilon)^* = -(D_{-\varepsilon})$. On the other hand, the centered operator D_c defined by $(D_c^s\psi)_i = (\psi_{i+e_s} - \psi_{i-e_s})/2\Delta v$ for $\quad s = 1,\dots,d$. The operators D_ε are clearly first-order approximations while D_c is a second-order one. For a given uncentered operator D_ε, the sets N^s are $N^s = \{i, i+\varepsilon_s e_s\}$. For the sake of simplicity, we restrict ourselves in the linear case (section 2) to discrete gradient $D = D_\varepsilon$ with any choice of ε. We shall also introduce the discretized divergence operator as $(D\cdot\psi)_i = \sum_{s=1}^d (D^s\psi)_i = \frac{1}{\Delta v}\sum_{s=1}^d \left(\varepsilon_s(\psi_{i+\varepsilon_s e_s} - \psi_i)\right.$.

2.2.2. The semidiscretized operator. The distribution function is approximated by a piecewise function on a fixed regular mesh of the form $v_i = i\Delta v$, where $i \in \mathbb{Z}^d$. Let us denote by $f_i(t)$ the value of the approximated distribution function at velocity v_i and time t. The evolution in time of these functions are governed by a coupled system of nonlinear equations which is the discretized version of (2.2):

$$\frac{df_i}{dt} = FP_i^L = (D^*\cdot p)_i, \quad p_i^s = g_i^s(D^s\log(f/M))_i, \tag{2.7}$$

where the terms g_i^s have now to be defined, M is the discretized Maxwellian with the same mass n_f as f, velocity u and temperature $T = 1$, as in (2.3). This Maxwellian is thus chosen such that its mass is the same as the mass of the initial distribution function:

$$\sum_i M_i = \sum_i f_i^0.$$

Note that the simplest and natural choice $g_i^s = f_i$ corresponds to the semidiscretization proposed by Degond and Lucquin in [25, 12] for the nonlinear FPL equation, described in the next section. In this paper, we shall use a modification of the scheme by defining, in a general way

$$g_i^s = \frac{(\sharp N^s)\prod_{k\in N^s} f_{i+k}}{\sum_{k'\in N^s}\left(\prod_{k\in N^s-\{k'\}} f_{i+k}\right)}, \quad i \in \mathbb{Z}^d, \tag{2.8}$$

where $(\sharp N)$ is the cardinal of any finite subset N of $\mathbb{Z}^d$. This term g_i^s is a rather good approximation of f_i when the distribution function is smooth but it is a rough one when the distribution function has some "hole," i.e., takes very small values at some velocity because all neighbors of these velocities with vanishing weight will be associated with vanishing g_i^s. We have the following estimates:

$$0 \le g_i^s \le (\sharp N^s) f_{i+k} \quad \forall i \in \mathbb{Z}^d \text{ and } \forall k \in N^s. \tag{2.9}$$

Using the choice $D = D_\varepsilon$ of discrete gradient with $(\sharp N^s) = 2$ and the correspondent definition of the set N^s, the scheme can be simplified further:

$$p_i^s = \frac{\varepsilon_s}{\Delta v} g_i^s\left(\log\left(\frac{f_{i+\varepsilon_s e_s}}{M_{i+\varepsilon_s e_s}}\right) - \log\left(\frac{f_i}{M_i}\right)\right)$$

for $s = 1, \dots, d$. Then, the system reads, surprisingly, in a form independent of the direction ε used to calculate the discrete gradient,

$$\frac{df_i}{dt} = FP_i^L = \frac{1}{\Delta v^2} \sum_{\mu \in \{-1,1\}} \sum_{s=1\dots d} g_{i,i+\mu e_s} \left(\log \left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} \right) - \log \left(\frac{f_i}{M_i} \right) \right), \tag{2.10}$$

with the terms $g_{i,j}$, which are given by

$$g_{i,j} = \frac{2 f_i f_j}{f_i + f_j} \quad \text{with} \quad g_i^s = g_{i,i+\varepsilon_s e_s} = \frac{2 f_i f_{i+\varepsilon_s e_s}}{f_i + f_{i+\varepsilon_s e_s}}, \quad g_{i-\varepsilon_s e_s}^s = g_{i,i-\varepsilon_s e_s}. \tag{2.11}$$

Hence, this modification ($f_i \mapsto g_i$) allows to symmetrize the formulation of the discretized linear Fokker–Planck equation.

2.2.3. Finite volume interpretation. Let us show that the scheme (2.10) can be derived using classical finite volume approach. Starting again from the FPL linear equation in its log form, i.e., (2.2) (again with $T = 1$)

$$\frac{\partial f}{\partial t} = \vec{\nabla}_v \cdot \left(f \vec{\nabla}_v \log(f/M_f) \right),$$

integrating it over the cell C_i defined as the centered cubic cell surrounding the point v_i and using the Green formula, one obtains

$$\int_{C_i} \frac{\partial f}{\partial t} dv = \sum_{\mu \in \{-1,1\}} \sum_{s=1,\dots,d} \int_{\partial C_{(i,i+\mu e_s)}} \mu f \frac{\partial \log f/M_f}{\partial v^s} d\sigma, \tag{2.12}$$

where $\partial C_{(i,i+\mu e_s)}$ stands for the interface between cells C_i and $C_{i+\mu e_s}$ (with a normal vector $\mu \vec{e_s}$ and, thus, a normal derivative $\mu \frac{\partial}{\partial v^s}$) and $d\sigma$ the superficial measure on this interface. By taking midpoint quadrature formula for each side of this equation, we have

$$\frac{df_i}{dt} = \frac{\Delta v^{d-1}}{\Delta v^d} \left(\sum_{\mu \in \{-1,1\}} \sum_{s=1,\dots,d} f\left(\frac{v_i + v_{i+\mu e_s}}{2} \right) \mu \frac{\partial \log \left(\frac{f}{M_f} \right)}{\partial v^s} \left(\frac{v_i + v_{i+\mu e_s}}{2} \right) + \mathrm{O}(\Delta v^2) \right). \tag{2.13}$$

Now, approximating $\partial \log f/M_f / \partial v^s (v_i + v_{i+\mu e_s}/2)$ with the second-order approximation

$$\frac{\mu}{\Delta v} \left(\log \left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} \right) - \log \left(\frac{f_i}{M_i} \right) \right),$$

and using a classical, second-order approximation designed for the numerical treatment of such diffusive equation [16] which consists of the harmonic averaged of the diffusion coefficients

$$f \left(v = \frac{v_i + v_{i+\mu e_s}}{2} \right) \approx \frac{2 f_i f_{i+\mu e_s}}{f_i + f_{i+\mu e_s}},$$

one recovers the scheme (2.10). Note that such approximation of the diffusion coefficients guarantees the continuity of the fluxes at the interface.

2.2.4. Reduction to a bounded velocity domain. From a numerical point of view, it is necessary to consider a bounded velocity domain, i.e., to assume that the index i belongs to a finite set of integer which is assumed of the form $\{1 \cdots n\}^d \stackrel{def}{=} I$ for the sake of simplicity. Let us define the "interior" set $\mathcal{I}$ such that $i \in \mathcal{I}$ if and only if $\forall k \in N$, $(i+k) \in I$. The definition (2.7) of FP_i^L is modified as follows on the discretized weak formulation:

$$\sum_{i \in I} \phi_i FP_i^L = -\sum_{i \in \mathcal{I}} \sum_{s=1,\dots,d} g_i^s (D\phi)_i \cdot (D \log(f/M_f))_i \,. \tag{2.14}$$

Then, FP_i^L is unchanged for any "interior" point $i \in \mathcal{I}$. For the "frontier" points, it leads to suppress the terms of the form

$$\left(\log\left(\frac{f_{i \pm \varepsilon_s e_s}}{M_{i \pm \varepsilon_s e_s}} \right) - \log\left(\frac{f_i}{M_i} \right) \right)$$

for any index such that $i \pm \varepsilon_s e_s \notin I$. Finally, FP_i^L, on a bounded domain I is of the form $\forall i \in I$

$$FP_i^L = \sum_{\mu \in \{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s} g_{i,i+\mu e_s} \left(\log\left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} \right) - \log\left(\frac{f_i}{M_i} \right) \right), \tag{2.15}$$

where the constant coefficients $a_{i,i+\mu e_s}$ were defined as $\frac{1}{\Delta v^2}$ if the point $i + \mu e_s$ lies in I, or 0 if not and $g_{i,i+\mu e_s}$ is given by (2.11).

2.2.5. The semidiscrete H-theorem. The H-theorem is satisfied with the following discrete entropy functional:

$$H = \sum_{i \in I} f_i \log(f_i/M_i). \tag{2.16}$$

Indeed, using the weak formulation (2.14) with $\phi_i = \log(f_i/M_i)$, we have using the mass conservation

$$\frac{dH}{dt} = -\sum_{i \in \mathcal{I}} \sum_{\mu \in \{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s} g_{i,i+\mu e_s} \left(\log\left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} \right) - \log\left(\frac{f_i}{M_i} \right) \right)^2 \leq 0.$$

Moreover, we have (if the terms g_i^s are strictly positive):

$$\frac{dH}{dt} = 0 \Leftrightarrow \forall (i,j) \in I^2, f_i/M_i = f_j/M_j. \tag{2.17}$$

Finally, using the mass conservation, we have $f_i = M_i \ \forall i \in I$.

2.3. Existence for the semidiscretized linear FPL equation. This section is devoted to the proof of the following theorem.

THEOREM 2.1. *Let $(f_i^0) \in \mathbb{R}^{(\sharp I)}$ such that $f_i^0 > 0 \ \forall i \in I$. Then, the Cauchy problem for the system (FP^L)*

$$\frac{df_i}{dt} = FP_i^L, \quad f_i(t=0) = f_i^0 \quad \forall i \in I,$$

with FP_i^L defined by (2.15) *is well posed. Moreover, there exists a global in time solution $(f_i(t))_{i\in I}$ such that*

$$\forall i \in I, \lim_{t\to\infty} f_i(t) = M_i, \tag{2.18}$$

where M_i is the associated discrete Maxwellian, defined by (2.3).

Proof. The local existence is obtained using standard Lipschitz property of the equations. Thus, there exists a solution at least for small time. The difficulty arises when some of the f_i tend to 0 due to the singularity of the log terms.

Once a positive lower bound is obtained for the f_i, one has an upper bound using the conservation of mass. Hence, there are only two alternatives: either there exists a global positive solution or the solution has a maximal lifetime $t_0 < \infty$ such that $f_i(t) \to 0$ when $t \to t_0$ for some $i \in I$. We shall now prove that the solution is global using a maximum principle. Let us recall the proof although it can probably be obtained using a more general theory, since it provides us an explicit lower bound for the distribution function. Define h_S and h_I as follows:

$$h_S(t) = \sup_{i\in I} \frac{f_i(t)}{M_i}, \quad h_I(t) = \inf_{i\in I} \frac{f_i(t)}{M_i}. \tag{2.19}$$

Note that h_I and h_S are continuous functions of t (for sufficiently small time t such that the solution exists) such that $h_S(0) \geq 1$ and $0 < h_I(0) \leq 1$ since the Maxwellian M_i has the same first moment as f_i. Let us prove that h_S is a decreasing function by contradiction. Assume there exists t_1 and $t_2 > t_1$ such that $h_S(t_2) > h_S(t_1)$. Then, there exists $t_3 \in [t_1, t_2]$ such that h_S is maximal. Let I_S be the set of indexes i for which $h_S(t_3) = f_i(t_3)/M_i$. At such point i, we have

$$\frac{d(f_i/M_i)}{dt}(t_3) \leq 0.$$

Indeed, the time evolution of f_i is governed by (2.15), and, setting $h_i = (f_i/M_i)$ is a sum of terms of the form $\log(h_{i+k}/h_i)(t_3)$, with $k \in N$, multiplied by positive terms $a_{i,i+k}g_{i,i+k}$. Thus, h_i being maximal ($i \in I_S$), these terms are negative. The minima cannot be isolated since in this case, we get $d(f_i/M_i)(t_3)\big/dt < 0$, which contradicts $i \in I_S$. Step by step, we obtain that $h_i = 1$, i.e., $f_i = M_i$ $\forall i \in I$. Finally, we have proved that h_S is decreasing. A similar proof holds for h_I increasing. This naturally provides a uniform bound for the weights f_i and for any time t

$$h_S(0) \sup_{i\in I} M_i \geq f_i(t) \geq h_I(0) \inf_{i\in I} M_i \quad \forall i \in I, \;\; \forall t \geq 0.$$

Let us now prove that $f_i \to M_i$ when $t \to \infty$. Indeed, the following "weighted L^2" distance between f_i and M_i

$$\tilde{D} = \sum_{i\in I} M_i \left(\frac{f_i}{M_i} - 1\right)^2,$$

is decreasing, using $\phi_i = (f_i/M_i - 1)$ in the weak formulation (2.14),

$$\begin{aligned}\frac{d\tilde{D}}{dt} = 2\sum_{i\in I} \frac{df_i}{dt}\left(\frac{f_i}{M_i} - 1\right) &= -2\sum_{i\in\mathcal{I}} \sum_{\mu\in\{-1,1\}} \sum_{s=1\cdots d} a_{i,i+\mu e_s} g_{i,i+\mu e_s} \left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} - \frac{f_i}{M_i}\right) \\ &\times \left(\log\left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}}\right) - \log\left(\frac{f_i}{M_i}\right)\right) \leq 0,\end{aligned}$$

using the standard inequalities $(x-y)(\log(x)-\log(y)) \geq 0$. The distance is decreasing and tends to 0 for any sequence of time. More precisely, the maximum principle, proved above, ensures the existence of $C>0$ (depending only on the initial data and particularly on the initial lower bound of the weights) such that

$$\frac{d\tilde{D}}{dt} \leq C \sum_{i\in\mathcal{I}} \sum_{\mu\in\{-1,1\}} \sum_{s=1\dots d} \left(\frac{f_{i+\mu e_s}}{M_{i+\mu e_s}} - \frac{f_i}{M_i}\right)^2 \leq \tilde{C}\tilde{D}.$$

Thus, the functions $f_i(t)$ tend exponentially to M_i $\forall i \in I$ when $t \to \infty$ using a Grönwall lemma. This last result can be related to a recent paper of Toscani [28] concerned with the continuous linear Fokker–Planck equation for which he obtains similar exponential decay toward the Maxwellian.

The convergence toward the equilibrium can also be proved using the Csiszar–Kullback inequality [19] for the continuous problem (see [28]). Since the terms f_i are bounded below for $t \in [0,\infty[$ then necessarily $\lim_{t\to\infty} H(t) = 0$. Applying the Csiszar–Kullback inequality which states that for any couple of strictly positive functions F and G and any probability measure dP we have

$$\| F - G \|^2_{L^1(dP)} \leq 2 \int F \ln\left(\frac{F}{G}\right) dP;$$

then

$$0 \leq \left(\sum_i |f_i - M_i|\right)^2 \leq 2H(f),$$

which implies that

$$\lim_{t\to\infty} \sum_i |f_i(t) - M_i| = 0. \qquad \square$$

Note that this theorem can be proved even in the case $g_i = f_i$. However, the following cannot be extended without the modification and the use of the estimate (2.9) on $g_{i,j}$ we have proposed.

2.4. The time-discretized linear FPL equation. Let us consider now the explicit time discretization of the preceding problem. The distribution function is assumed known at time t and equal to $(f_i)_{i\in I}$. Let us denote by $\bar{v}$ the computed value of any variable v at time $t+\Delta t$. The scheme is of the form

$$\bar{f}_i = f_i + \Delta t FP_i^L, \tag{2.20}$$

where FP_i^L is defined by (2.15). The main properties of this explicit in time scheme (2.20) are summarized in the following proposition.

PROPOSITION 2.2. *Let $(f_i^0) \in \mathbb{R}^{(\sharp I)}$ such that $f_i^0 > 0$ $\forall i \in I$. Then there exists a time step of the form $C/\Delta v^2$ with the constant C depending only on the initial condition for which the scheme* (2.20) *is positive and conservative and decays entropy. Furthermore, the scheme leads to a solution which verifies*

$$\forall i \in I, \lim_{t\to\infty} f_i(t) = M_i, \tag{2.21}$$

where M_i is the associated discrete Maxwellian, defined by (2.3).

The proof is postponed until the appendix. Note that the scheme leads to a nonoscillatory solution since the maximum principle prevents the creation of any supplementary local extremum (since local maximum decreases and local infimum increases).

3. The nonlinear FPL equation. We shall now consider the tridimensional nonlinear FPL equation in the homogeneous case for any interaction potential. We restrict ourselves to a single-species plasma since the methods can easily be extended to the multispecies case as explained in [7].

3.1. The continuous nonlinear FPL equation. We denote by $f(v,\, t)$ the distribution function, a solution of the following scaled FPL equation:

$$\frac{\partial f}{\partial t} = Q(f,f) = \nabla_v . \left(\int_{\mathbb{R}^3} \Phi(v-v_*)((\vec{\nabla}_v f) f_* - (\vec{\nabla}_{v_*} f) f) dv_* \right), \tag{3.1}$$

where $Q(f,\, f)$ is the FPL collision operator written in the so-called Landau form with the standard notations (for example, $f_* = f(x, v_*, t)$) and $\Phi(v)$ is the following 3×3 matrix:

$$\Phi(v) = |v|^{\gamma+2} S(v), \quad S(v) = I_3 - \frac{v \otimes v}{|v|^2}. \tag{3.2}$$

$S(v)$ is the orthogonal projector onto the plane orthogonal to v. γ is a real parameter which leads to the usual classification in hard potentials ($\gamma > 0$), Maxwellian molecules ($\gamma = 0$) or soft potentials ($\gamma < 0$). This latter case involves the Coulombian case (i.e., $\gamma = -3$) which is of primary importance for plasma applications. The well-known physical properties of the FPL operator are similar to that of the Boltzmann operator such as the decay of the entropy, the conservation of mass, momentum, and energy, and the characterization of the equilibrium states by Maxwellians. These properties can easily be shown on the weak form of the FPL operator which can be found in [11, 7, 25, 12].

3.2. The semidiscretized nonlinear FPL equation. The distribution function is approximated by piecewise constant functions on a fixed regular mesh of the form $v_i = i\Delta v$, where $i = (i_1, i_2, i_3)$ belongs, as in the linear case, to a finite set of integer which is assumed of the form $\{1 \cdots n\}^3 \stackrel{def}{=} I$ for the sake of simplicity. We refer to [11, 25, 12] for a detailed derivation of this discretization. Denote by $f_i(t)$ the value of the approximated distribution function at velocity v_i and time t. The time evolution of this discretized function is then governed by a coupled system of nonlinear equations of the form (for $i \in I$)

$$\frac{\partial f_i}{\partial t} = (D^*.p)_i, \quad p_i = \Delta v^3 \sum_{j \in \mathcal{I}} f_i f_j \Phi\left((i-j)\Delta v\right) \left((D \log f)_i - (D \log f)_j\right), \tag{3.3}$$

where D is again a finite difference operator that approximates the usual gradient operator ∇ at least up to the first order, D^* its formal adjoint, as defined in section 2.2.1. The reduction to a bounded domain follows the same lines as in the linear case presented before. In the weak formulation, this system of differential equations can be written equivalently for any test sequence $(\psi_i)_{i \in I}$ as

$$\begin{aligned} &\sum_{i \in I} \frac{\partial f_i}{\partial t} \psi_i \Delta v^3 \\ &\quad = -\frac{1}{2} \Delta v^6 \sum_{(i,j) \in \mathcal{I}^2} f_i f_j \left((D\psi)_i - (D\psi)_j\right)^T \Phi(v_i - v_j) \left((D(\log f))_i - (D(\log f))_j\right). \end{aligned} \tag{3.4}$$

As shown in [11, 12, 25], this discrete model is conservative and decays entropy (provided that there exists a solution, which is the aim of the present paper). The only

equilibrium states are the discrete Maxwellians for decentered approximation D_ε of the gradient but introduce spurious collisional invariant for the centered D_c one (previously defined in the linear case). The good choice for the discretized Fokker–Planck operator is the arithmetic average over all (eight) decentered operators D_ε which is equal to the centered operator plus a diffusive perturbation (see [7] for details).

From the positivity of the matrix Φ, it is easy to check that such a scheme decays the entropy using the weak form (3.4) with $\psi_i = \log f_i$

$$H(t) \stackrel{def}{=} \Delta v^3 \sum_{i\in I} f_i(\log f_i) \le \Delta v^3 \sum_{i\in I} f_i^0(\log f_i^0) = H(0). \tag{3.5}$$

Note that this property is not achieved if one discretizes the Fokker–Planck operator in the "nonlog" form (see [12] for a precise statement). This was the first reason for using the "log"-discretization we presented here. There is another reason which is also of main importance: there exists some positive initial condition for which the "nonlog" discretized FPL equations leads to a solution with negative values of the distribution function in arbitrary small time (see appendix for details).

3.3. Existence for a semidiscretized nonlinear FPL equation. We shall now turn to the proof of existence of positive solutions for the Cauchy problem for the system defined by (3.3) with positive initial conditions $f_i(t=0) = f_i^0 > 0\ \forall i \in I$. We also refer to [7] for methods such as multigrid or sublattice for reducing the numerical cost in evaluating this quadratic operator. These methods are based on a simplified discrete operator of the form

$$\begin{aligned}(3.6)\quad &\Delta v^6 \sum_{i\in\mathbb{Z}^3} FP_i \psi_i \Delta v^3 \\ &= -\frac{1}{2} \sum_{(i,j)\in I^2} a_i a_j \left((D\psi)_i - (D\psi)_j\right)^T \Phi(v_i - v_j)\left((D\log f)_i - (D\log f)_j\right),\end{aligned}$$

where the coefficients a_i depend on the distribution function, some of them can be null, and their sum over all the indices is bounded. The number of nonnull terms determines the cost of the method. For the original discrete operator there are $(\sharp I)^2$ nonnull terms so the method is quadratic. In multigrid methods, since Monte Carlo integration is used [7], only $\sharp I \log(\sharp I)$ terms are nonzero at each time step. Therefore the cost is only of the order of $\sharp I \log(\sharp I)$. The following analysis also applies for such rapid methods. The existence of solutions for small time can be easily obtained using classical Cauchy–Lipschitz theorem. Indeed, there is no singularity in this system neither in the log terms, using $f_i^0 > 0$, nor in the matrix Φ, since for $i \neq j$ we have $\|v_i - v_j\| \ge \Delta v$.

Then, these solutions had to remain positive due to the log terms and they are global in time provided that none of the f_i vanishes. Indeed, we get for free an upper bound (uniform in time, depending on Δv) for f_i using the conservation of mass $\sum_{i\in I} f_i \Delta v^3 = \sum_{i\in I} f_i^0 \Delta v^3$. Finally, we have the only two following alternatives: either there exists a maximal time $t_0 > 0$ of existence, i.e., such that for at least some index $i_0 \in I$, $\lim_{t\to t_0} f_{i_0}(t) = 0$, or there exists a solution global in time. In other words, existence of global solution for the differential equations is related to the fact that f_i can only vanish in infinite time.

Assuming that there exists only one index i_0 for which the distribution tends to 0 when $t \to 0$, one separates in the various terms involved in $(D^*.p)_i$ the ones containing

$\log(f_{i_0})$ which blows up. Then, it is easy to check that this term is multiplied by a negative constant and therefore, the leading order term in df_{i_0}/dt is positive which provides the contradiction. However, this method cannot be extended, at least to our knowledge, to the general case where the distribution function vanishes simultaneously at several locations.

Showing that the weights f_i cannot vanish in finite time is equivalent to showing that the function

$$K = \sup_{i\in\mathcal{I},k\in N} \left|\frac{f_i}{f_{i+k}}\right| \tag{3.7}$$

remains bounded in finite time as in the linear case. This function is convenient since these ratios actually appear in the $D(\log f)_i$ terms. A direct calculation, which is left to the reader and related to the one given in the next section, gives the following estimates:

$$\left|\frac{dK}{dt}\right| \leq CK^2 \log(K) \tag{3.8}$$

for some constant $C > 0$. This estimate obviously does not give an upper bound for the value of K for arbitrary large time and therefore, no lower bound for the f_i in finite time. Other methods for proving the existence of positive solution for the system of ODEs (3.3) rely on functional approaches (i.e., find a function of f_i which tends to ∞ if one of the $f_i \to 0$ and proves that it remains bounded) like, for example, the Linnick functional [4]. Note that for the continuous Boltzmann or FPL equations, the distribution function can be bounded below by some Maxwellian [29, 14, 15].

To ensure the existence of global positive solutions, we introduce a slightly different discretized operator which has exactly the same properties (conservation, entropy) as the one defined in [11, 7, 25, 12] but for which the proof can be complete using the first direct method. The trick consists of modifying the terms f_i and f_j, respectively, in the formulae (3.3) by some approximations g_i and g_j, respectively, defined as in the linear case by

$$g_i = \begin{cases} \dfrac{(\sharp N)\prod_{k\in N} f_{i+k}}{\sum_{k'\in N}\left(\prod_{k\in N-\{k'\}} f_{i+k}\right)}, & i\in\mathcal{I}, \\ 0 \text{ if } i\in I-\mathcal{I}, \end{cases}$$

where N is a finite subset of $\mathbb{Z}^3$ and $(\sharp N)$ is the cardinal of N. Let us recall that g_i is a rather good approximation of f_i when the distribution function is smooth but is rough when the distribution function is very peaked at some velocity. The Cauchy problem associated with the modified ODEs, i.e., the semidiscretized FPL equation (3.3) reads for $i \in I$

$$\frac{\partial f_i}{\partial t} = FP_i = (D^*.p)_i, \quad p_i = \Delta v^3 \sum_{j\in\mathcal{I}} g_i g_j \Phi((i-j)\Delta v)\left((D\log f)_i - (D\log f)_j\right), \tag{3.9}$$

We have the following result.

THEOREM 3.1. *Let $(f_i^0) \in \mathbb{R}^{(\sharp I)}$ such that $f_i^0 > 0$ $\forall i \in I$. The Cauchy problem with initial conditions $f_i(t=0) = f_i^0$ for the differential system* (3.9) *has a unique positive entropy solution for arbitrary large time.*

Proof. The existence and unicity of the solution of system (3.9) with strictly positive initial data for small time is obtained using classical Cauchy–Lipschitz theorem.

We will now prove that the solution cannot vanish at some velocities in finite time and, thus, the solution is global in time. The discrete H-theorem is given by (3.5) and can be proved using the weak formulation. We recall the following estimates:

$$0 \le g_i \le (\sharp N) f_{i+k} \quad \forall i \in \mathcal{I} \text{ and } \forall k \in N. \tag{3.10}$$

Using definition of K and p_i, i.e., (3.7) and (3.9), we have the following estimate for the vectors p_i:

$$\|p_i\| \le C g_i \log(K), \tag{3.11}$$

where C is a generic constant throughout the rest of the paper, depending on the number of grid points ($\sharp I$), the potential parameter γ, the velocity mesh size Δv, the initial condition $(f_i^0)_{i\in I}$, the coefficients a_k, and the cardinal of the set N. Indeed, we have the following upper bounds:

$$\sup_{(i,j)\in I^2} \|\Phi\,(v_i - v_j)\| \le C, \quad |f_i| \le C \quad \forall i \in I,$$
$$\|(D \log f)_i\| \le C \quad \forall i \in \mathcal{I}.$$

Then, using (3.10), we have

$$|(D^*.p)_i| \le C \log(K) \sup_{k\in N} g_{i-k} \le C \log(K) f_i. \tag{3.12}$$

Then, we have for any $i \in \mathcal{I}$ and any $k \in N$

$$\frac{d(f_i/f_{i+k})}{dt} = \frac{1}{f_{i+k}} \frac{df_i}{dt} - \frac{f_i}{f_{i+k}^2} \frac{df_{i+k}}{dt}.$$

Finally, using (3.12) and since $\dfrac{df_i}{dt} = (D^*.p)_i$, we have

$$\left| \frac{dK}{dt} \right| \le C K \log(K), \tag{3.13}$$

which implies $K(t) \le K(0) \exp(\exp(Ct))$ and this concludes the proof. $\square$

This proof can be carried out for the diffusive perturbation defined in [7] by again changing the f_i into g_i as explained above. We do not detail the proof since it follows exactly the same lines. The key point is that the modified scheme (with g_i defined by (2.8)) satisfies (3.10), which implies (3.12), while the original scheme [11, 7, 25, 12] (with $g_i = f_i$) leads to

$$|(D^*.p)_i| \le C K \log(K) f_i, \tag{3.14}$$

and thus (3.13) becomes (3.8).

Note that for large time, the distribution function should tend at least formally to the discretized Maxwellian since its entropy decays (see Csiszar–Kullback inequality). Therefore, there must exist some constant $K_0 > 1$ such that $K(t) \le K_0$ for all time. However, the convergence toward the discrete Maxwellian or equivalently the existence of a lower bound for the weights f_i uniform in time remains an open problem up to now for the discretized nonlinear FPL equation. This point is currently being investigated.

3.4. The time-discretized nonlinear FPL equation. Let us consider an explicit time discretization as in the linear case. The distribution function is assumed known at time t and equal to f_i and its value at time $\bar{t} = t + \Delta t$ denoted by $\bar{f}_i$ is given by the following explicit scheme:

$$\bar{f}_i = f_i + \Delta t FP_i, \tag{3.15}$$

where FP_i is defined by (3.9). We shall determine conditions on the time step Δt under which the scheme gives positive and entropy solution for an arbitrary large time. We resume our results in the following proposition.

PROPOSITION 3.2. *There exists a time-step sequence Δt_n such that the scheme* (3.15) *defines recursively a positive and entropy solution at any time (i.e., $\sum \Delta t_n = \infty$).*

Proof. First, note that (3.12) gives a positive constant $C > 0$ such that

$$|FP_i| = |(D^*.p)_i| \leq C \log(K) f_i,$$

where $K = \max_{i \in \mathcal{I}, k \in N} f_i / f_{i+k}$. Let us define $\Delta t_1 = 1/C \log K$ and choose $\Delta t = \alpha \Delta t_1$, with $0 < \alpha < 1$. Then, we have using (3.15) and (3.12),

$$\bar{K} = \max_{i \in \mathcal{I}, k \in N} \frac{\bar{f}_i}{\bar{f}_{i+k}} \leq \frac{K(1+\alpha)}{(1-\alpha)} = \beta K,$$

with $\beta = (1+\alpha)/(1-\alpha) > 1$. Note the difference with the linear case where the maximum principle insures that the function K decreases at each iteration. Thus, we have by recursion at iteration n

$$\log(K_n) \leq n \log(\beta) + \log(K_0),$$

with $K_0 = K(0) = \max_{i \in \mathcal{I}, k \in N} f_i^0 / f_{i+k}^0 < \infty$. Therefore, for the time step Δt_n at iteration n, which is defined recursively by $\Delta t_n = \alpha / C \log(K_n)$ and for which the solution is positive, we have the following estimate:

$$\Delta t_n \geq \frac{\alpha}{C(n \log(\beta) + \log(K_0))}.$$

The right-hand side of this inequality leads to a divergent sum and thus the solution remains positive and can be constructed for any arbitrary large time, i.e., after n iterations the time is equal to

$$t_n = \sum_{k \leq n} \Delta t_k \to \infty, \quad n \to \infty.$$

We have now to check that the entropy decays. Defining the discrete entropy H by (3.5), we obtain, as in the linear case (see appendix A-1), that it is decreasing provided that the time-step is smaller than

$$\Delta \tilde{t}_1 \stackrel{def}{=} \min \left(\alpha \Delta t_1, \frac{-\Delta v^3 \sum_{i \in I} FP_i \log(f_i)}{\Delta v^3 \sum_{i \in I} \dfrac{(FP_i)^2}{f_i}} \right).$$

Using the weak formulation, we have

$$\left| \Delta v^3 \sum_{i \in I} FP_i \log(f_i) \right| = \Delta v^6 \sum_{(i,j) \in \mathcal{I}^2} g_i g_j X_{i,j}^T \Phi(v_i - v_j) X_{i,j}, \tag{3.16}$$

where $X_{i,j}$ is a d vector defined by $X_{i,j} = (D \log f)_i - (D \log f)_j$. On the other hand, using the convention where terms of the form a_{i+k} are set to zero if $i+k \notin I$, we have for any $i \in I$

$$(FP_i)^2 = (D^*.p)_i^2 \leq \frac{C'}{\Delta v^2} \sum_{k \in N} \|p_{i+k}\|^2 ,$$

where C' is a constant which depends only on the cardinal of the set N and

$$\|p_{i+k}\|^2 = \left\| g_{i+k} \Delta v^3 \sum_{j \in \mathcal{I}} g_j \Phi(v_{i+k} - v_j) X_{i+k,j} \right\|^2 .$$

Let us introduce the sequences U_j and V_j as

$$U_j = \left(\sqrt{\frac{g_{i+k} g_j \Delta v^3}{\|v_{i+k} - v_j\|^s}} \right)_j , \quad V_j = \left(\sqrt{\frac{g_{i+k} g_j \Delta v^3}{\|v_{i+k} - v_j\|^s}} \, \|S(v_{i+k} - v_j) X_{i+k,j}\| \right)_j .$$

The definition of p_{i+k} leads, using the Cauchy–Schwarz inequality and definition of the matrix $S(v)$ (with $s = 1$ in the Coulombian case), to

$$\|p_{i+k}\|^2 \leq \|U\|^2 \|V\|^2 ,$$

with the norm

$$\|U\|^2 = \sum_{j \in \mathcal{I}} \frac{g_{i+k} g_j \Delta v^3}{\|v_{i+k} - v_j\|^s}, \ \|V\|^2 = \Delta v^3 \sum_{j \in \mathcal{I}} g_{i+k} g_j \|v_{i+k} - v_j\|^s \|\Phi(v_{i+k} - v_j) X_{i+k,j}\|^2 .$$

Thus, we have, using twice $g_i \leq (\sharp N) f_i \ \forall i$,

$$\begin{aligned} &\|p_{i+k}\|^2 / f_i \\ &\leq (\sharp N)^2 \left(\sup_{i \in I} \sum_{j \in \mathcal{I}} \frac{f_j \Delta v^3}{\|v_i - v_j\|^s} \right) \Delta v^3 \sum_{j \in \mathcal{I}} g_{i+k} g_j \|v_{i+k} - v_j\|^s \|\Phi((i+k-j)\Delta v) X_{i+k,j}\|^2 , \end{aligned}$$

and adding these inequalities, we obtain

$$\begin{aligned} &\sum_{i \in I} \frac{(FP_i)^2}{f_i} \Delta v^3 \\ &\leq \frac{C'}{\Delta v^2} (\sharp N)^2 \left(\sup_{i \in I} \sum_{j \in \mathcal{I}} \frac{f_j \Delta v^3}{\|v_i - v_j\|^s} \right) \Delta v^6 \sum_{(i,j) \in \mathcal{I}^2} g_i g_j \|v_i - v_j\|^s \|\Phi(v_i - v_j) X_{i,j}\|^2 . \end{aligned}$$

Moreover, we have for all vectors v and X

$$X^T S(v) X = \|S(v) X\|^2 \ \Rightarrow \ X^T \Phi(v) X = \|v\|^s \|\Phi(v) X\|^2 ,$$

and then, using (3.16), we have proved that

$$\sum_{i \in I} \frac{(FP_i)^2 \Delta v^3}{f_i} \leq \frac{C'}{\Delta v^2} (\sharp N)^2 \left(\sup_{i \in I} \sum_{j \in \mathcal{I}} \frac{f_j \Delta v^3}{\|v_i - v_j\|^s} \right) \times \Delta v^3 \sum_{i \in I} FP_i \log(f_i),$$

and thus

$$\Delta t \stackrel{def}{=} \min\left(\alpha\Delta t_1, \frac{\Delta v^2}{C'(\sharp N)^2} \times \left(\sup_{i\in I}\sum_{j\in\mathcal{I}}\frac{f_j\Delta v^3}{\|v_{i+k}-v_j\|^s}\right)^{-1}\right)$$

yields a positive and entropy scheme. ◻

For the simplest uncentered discrete gradient, a simple calculus gives us $C' = 4$ and $(\sharp N) = 4$. For other choices of the discrete gradient the calculus of these constants can be achieved by straightforward calculations. In numerical simulations, the time step that guarantees positivity and entropy decay of the solution tends to a constant one (and not to zero as the time-step sequence constructed in the proof).

Moreover, the condition on the entropy is more restrictive than the positivity one and insures a nonoscillatory solution. This differs from the linear case. If the condition on the entropy is relaxed, it is readily seen on numerical simulations (see, for example, [6]) that unphysical oscillations appear.

4. Conclusions. The existence of a global, positive, conservative, and entropy solution for the semidiscretized nonlinear FPL equation as proposed (on Landau-log form) in [12, 3, 27, 5, 26] cannot be proved, at least to our knowledge, without the modification proposed here (i.e., $f_i \mapsto g_i$). Moreover, when such velocity discretized models are also discretized in time, it is not proved that they still preserve entropy decay and positivity of the solution even though these properties are satisfied by the semidiscretized models [11].

We show that the constructed discrete model for FPL equation gives global positive solution and its time-discretized version preserves all the properties of the semidiscretized one. In the linear case, the modification that we propose is related to a classical approximation of the diffusion coefficients. For the nonlinear equation, we give sufficient, CFL-like conditions on the time step that guarantees entropy decay and positivity.

We refer to [7, 22] for fast algorithms for solving the nonlinear Fokker–Planck equation in Landau form. As explained in the present paper, the proofs apply to the multigrid and sublattice methods and can probably be extended to multipole methods using the modification we propose. Let us also mention a forthcoming paper concerned with the isotropic distribution function [6], for which the results presented here can be improved. It is possible in this particular case to use a nonuniform grid, to obtain a uniform estimate on the time step (instead of the divergent series we have constructed), and to reduce the quadratic cost to a linear cost without any supplementary approximation. The isotropic case can be used to compute reference solutions—or benchmarks—since the linear cost of this one-dimensional problem makes it possible to compute very accurate solutions even in the Coulombian case ($\gamma = -3$), for which there is no known explicit solution unlike the Maxwellian case ($\gamma = 0$, see [21]).

The next step is to prove the convergence of the constructed sequences of approximated solutions toward the solution of the continuous FPL equation. First, one shall study the convergence of the time-discretized solution (using the explicit scheme presented in section 3.4) toward the solution of the system of ODE corresponding to the semidiscretized FPL equation presented in section 3.2 when $\Delta t \to 0$. Second, we shall prove convergence of the solution of the semidiscretized FPL equation to the solution of the continuous equation when $\Delta v \to 0$. This result is much more difficult to prove.

Appendix. A1. Proof of Proposition 2.2. Let us define $h_i = f_i/M_i$. We have from (2.20)

$$\bar{h}_i = h_i + \frac{FP_i^L}{M_i}\Delta t h_i \left(1 + \Delta t \sum_{\mu\in\{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s} \frac{g_{i,i+\mu e_s}}{f_i} \log\left(\frac{h_{i+\mu e_s}}{h_i}\right)\right).$$

Recall that h_I, h_S defined in the preceding proof can be equivalently written as

$$h_I = \inf_{i\in I, k\in N|(i+k)\in I} h_{i+k}, \quad h_S = \sup_{i\in I, k\in N|(i+k)\in I} h_{i+k}.$$

We also define $K = h_S/h_I > 1$. Let us prove first that there exists a uniform lower bound on the time step Δt for which h_S is decreasing and h_I is increasing. Assume

$$\Delta t \le \Delta t_1 \stackrel{def}{=} \frac{1}{M} \inf_{x\in]1,K]} \frac{(x-1)}{x\log x} < \infty,$$

with (using the definitions of $a_{i,i+\mu e_s}$)

$$M = \frac{8d}{\Delta v^2} \ge 4 \sum_{\mu\in\{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s}.$$

Using (2.9) we have $g_{i,i+\mu e_s} \le 2f_i$ by construction and therefore

$$h_i + 2\Delta t \left(\sum_{\mu\in\{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s}\right) h_i \log\left(\frac{h_I}{h_i}\right) \le \bar{h}_i,$$

and

$$\bar{h}_i \le h_i + 2\Delta t \left(\sum_{\mu\in\{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s}\right) \; h_i \log\left(\frac{h_S}{h_i}\right).$$

Moreover, for $\Delta t \le \Delta t_1$, we have

$$4\Delta t \left(\sum_{\mu\in\{-1,1\}} \sum_{s=1,\dots,d} a_{i,i+\mu e_s}\right) \; x\log x \le x - 1 \quad \forall x \in]1,K].$$

This implies (for $x = h_i/h_I$ and $x = h_S/h_i$) that

$$h_I \le (h_I + h_i)/2 \le \bar{h}_i \le (h_i + h_S)/2 \le h_S.$$

Thus, $\sup_{i\in I} h_i$ decreases, $\inf_{i\in I} h_i$ increases, and $\bar{K} \le K$. This discrete maximum principle insures that the scheme is positive. These bounds for the time-steps are not optimal. Since the coefficients $a_{i,i+k}$ depend on the velocity mesh size as Δv^{-2}, the stability condition we get, can be written in the form

$$\Delta t \le \Delta t_1 = C\Delta v^2, \quad C = \frac{1}{8d}. \inf_{x\in]1,K(0)]} \frac{(x-1)}{x\log x}$$

as is usual for such convection-diffusion operator.

We shall determine another condition for the scheme being entropy decaying. Let us assume the first condition is satisfied. Thus, the terms f_i and also $g_{i,i+\mu e_s}$ have a uniform lower and upper bound. The entropy at time $t+\Delta t$ is of the form

$$\bar{H} = \sum_{i\in I}(f_i + \Delta t FP_i^L)\log\left((f_i + \Delta t FP_i^L)/M_i\right).$$

Then, using $\log(1+h) \leq h\ \forall h > -1$, we have (with $h = \frac{\Delta f}{f} \geq -1$ since the scheme is positive)

$$\begin{aligned}(f+\Delta f)\log\left(\frac{f+\Delta f}{M}\right) &= (f+\Delta f)\log\left(\frac{f}{M}\left(1+\frac{\Delta f}{f}\right)\right)\\ &\leq (f+\Delta f)\left(\log\left(\frac{f}{M}\right)+\frac{\Delta f}{f}\right)\\ &\leq f\log\left(\frac{f}{M}\right)+\Delta f\log\left(\frac{f}{M}\right)+\Delta f+\frac{\Delta f^2}{f}.\end{aligned}$$

Summing this inequality (with $f = f_i$ and $\Delta f = \Delta t FP_i^L$) over $i \in I$, the first term of the right-hand side gives H, the third vanishes by mass conservation, and one obtains

$$\bar{H} \leq H + \Delta t \sum_{i\in I} FP_i^L \log(f_i/M_i) + (\Delta t)^2 (FP_i^L)^2/f_i \stackrel{def}{=} \tilde{H}.$$

Note that, as previously shown, we have if $f_i \neq M_i$ for at least one index $i \in I$

$$\sum_{i\in I} FP_i^L \log(f_i/M_i) < 0.$$

Therefore, the scheme is entropy decaying provided that the time step verifies

$$\Delta t \leq \Delta t_2 \stackrel{def}{=} \min\left(\Delta t_1, \frac{-\sum_{i\in I} FP_i^L(\log(f_i/M_i)}{\sum_{i\in I}(FP_i^L)^2/f_i}\right) < \infty.$$

By the definition (2.15) of FP_i^L, and applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned}(FP_i^L)^2 \leq &\left(\sum_{\mu\in\{-1,1\}}\sum_{s=1,\dots,d} a_{i,i+\mu e_s} g_{i,i+\mu e_s}\right)\\ &\times\left(\sum_{\mu\in\{-1,1\}}\sum_{s=1,\dots,d} a_{i,i+\mu e_s} g_{i,i+\mu e_s}\log^2\left(\frac{h_{i+\mu e_s}}{h_i}\right)\right).\end{aligned}$$

The first term of the right-hand side is less than $2f_i \sum_{\mu\in\{-1,1\}}\sum_{s=1,\dots,d} a_{i,i+\mu e_s}$ using again (2.9). The second term already appears in the computation of $\frac{dH}{dt}$. Dividing this inequality by f_i, using the definition $a_{i,i+\mu e_s}$, and summing over the indices i give

$$\sum_{i\in I}\frac{(FP_i^L)^2}{f_i} \leq \frac{4d}{\Delta v^2}\left(-\sum_{i\in I} FP_i^L(\log(f_i/M_i)\right).$$

We set $\delta t = \frac{\Delta v^2}{4d}$. Note that $\delta t \geq 2\Delta t_1$ since

$$\inf_{x\in]1,K(0)]} \frac{(x-1)}{x\log x} \leq 1.$$

Then, any Δt smaller than Δt_1 yields to a positive, entropy, and nonoscillatory scheme. Note that the entropy condition is less restrictive than the positivity one in the linear case. The scheme (2.20) defines a sequence of $\{f_i^n\}_{n\in\mathbb{N}}$, which serves us to define a piecewise-constant in time function and v by

$$f_i(t) = f_i^n \quad \forall t \in [t_n, t_{n+1}[,$$

with $t_0 = 0$, and $t_{n+1} = t_n + \Delta t$ and the f_i^n defined above by recursion. We have

$$K^{-1}M_i \leq f_i \leq KM_i \quad \forall i \in I,$$

with $K = K(0)$, defined above. We have now to prove that this discrete solution tends to the Maxwellian. First, note that the associated entropy H defined by

$$H(t) = \sum_{i\in I} f_i^n \log(f_i^n/M_i) \quad \forall t \in [t_n, t_{n+1}[$$

is decreasing and has a lower bound. Thus, it converges and the Cauchy criteria implies that $(H_{n+1} - H_n) \to 0$ as $n \to \infty$ where we have set $H_n = H(t_n)$. Define $\tilde{H}$ for $t \in [t_n, t_{n+1}[$ as previously by

$$\tilde{H}(t) = H(t_n) + (t-t_n)\sum_{i\in I} FP_i^L \log(f_i^n/M_i) + (t-t_n)^2 (FP_i^L)^2/f_i^n.$$

Then, $\tilde{H}$ satisfies

$$\tilde{H}(t_n) = H(t_n), H(t) - H_n \leq \tilde{H}(t) - H_n \leq 0 \quad \forall t \in [t_n, t_{n+1}[.$$

Thus, we have for $\Delta t = t_{n+1} - t_n \leq \Delta t_1 \leq \delta t/2$ using $(H_{n+1} - H_n) \to 0$:

$$\Delta t_1 \sum_{i\in I} (FP_i^L)^2/f_i + \sum_{i\in I} FP_i^L \log(f_i^n/M_i)) \to 0,$$

and, since we have proved

$$\left(-\sum_{i\in I} FP_i^L \log(f_i^n/M_i)\right) \geq (\delta t)\left(\sum_{i\in I} (FP_i^L)^2/f_i^n\right),$$

we obtain $\sum_{i\in I} FP_i^L \log(f_i^n/M_i)) \to 0$ when $n \to \infty$ which is equivalent to $f_i^n \to M_i$ $\forall i \in I$ since the g_i have a uniform lower bound. As for the semidiscretized model, the convergence of the sequence f_i^k toward the equilibrium can also be proved using the Csiszar–Kullback inequality and the fact that under the time-step restrictions to ensure the decay of the entropy and the positivity of the distribution function we have necessarily $\lim_{k\to\infty} H^k = 0$.

This concludes the proof. $\square$

A2. Negative solution for the nonlog form of nonlinear FPL equation. The nonlog formulation of the discretized nonlinear FPL equation (3.3) can be written as

$$\frac{\partial f_i}{\partial t} = (D^*.p)_i, \quad p_i = \Delta v^3 \sum_{j \in \mathcal{I}} \Phi\left((i-j)\Delta v\right)\left(f_j(Df)_i - f_i(Df)_j\right), \quad i \in I.$$

Let us consider, for example, $n = 8$, and the following initial data: $f_i = \mu \; \forall i \in I$ except for two indices $i_0 = (2,2,2)$ and $j_0 = (6,6,6)$ where $f_{i_0} = f_{j_0} = 1$. We choose for the log form the simplest centered operator, i.e., D_c. First, let $\mu = 0$. The system defined above gives a negative value for the derivative of f_k with respect to t at time $t = 0$ for $k = (7,6,7)$. A numerical computation of the terms p_i gives

$$\frac{d}{dt} f_k(t=0) = -1.457165e^{-04}.$$

Therefore, a positive initial condition gives a solution such that $\exists k \in I$, $f_k(t) < 0$ for any arbitrary small value of t. Finally, the continuity of the solution of the nonlog formulation of FPL equation with respect to the initial data implies that, for μ small enough, there exists t_μ such that $f_k(t_\mu) < 0$ for some indices k. Thus, a strictly positive initial data becomes negative after finite time (t_μ) when using the nonlog form of the FPL equation. On the contrary, this proof does not apply to the log form which is singular when some f_k tends to 0 and, thus, its solutions do not depend continuously on the initial data near such situations (with vanishing weights). Numerical simulations illustrating this property are available by sending e-mails to the authors.

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