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NUMERICAL SOLUTION OF AN IONIC FOKKER–PLANCK EQUATION WITH ELECTRONIC TEMPERATURE*

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Abstract. We describe a numerical scheme for dealing with an ion/electron collision operator of the Fokker–Planck type; for that purpose, we introduce the notion of the *entropic average* of two positive quantities. This scheme has the property to be entropic in the sense of Boltzmann's H-theorem under a *CFL* criteria. Moreover, we prove that the solution of the semidiscrete scheme converges towards a unique Maxwellian equilibrium state when the time grows. Numerical applications are given and show that our scheme is more precise than the classical Chang–Cooper one.

Key words. kinetics model, Fokker–Planck–Landau equation, plasma physics, numerical scheme

AMS subject classifications. 65M06, 65M12, 82C40, 82D10

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Introduction. In hot plasmas such as the plasmas of microballs in the inertial confinement fusion framework (see [1]), the characteristic length of variation of hydrodynamic quantities may be of the order of a micrometer, but when the temperature arises up to some KeV, the mean free path of the ions may be greater than a micrometer. Thus, to simulate the behavior of such a plasma, it is necessary to study the kinetic models of the evolution of ions coupled with an electronic population which is assumed to be Maxwellian. Before writing the relevant kinetic model, we recall the most simple fluid model for hot plasmas which is called *two-temperatures Euler equations* (in what follows, we assume that there is only one ionic species whose atomic mass is m, the ionization level being Z):

$$\frac{\partial}{\partial t}N + \nabla_x \cdot (N\overrightarrow{U}) = 0, \tag{0.1}$$

$$\frac{\partial}{\partial t}(mN\overrightarrow{U}) + \nabla_x \cdot (mN\overrightarrow{U} \otimes \overrightarrow{U}) + \nabla_x(NT + P_e) = \overrightarrow{0}, \tag{0.2}$$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}NT\right) + \nabla_x\left(\frac{3}{2}NT\overrightarrow{U}\right) + NT\nabla_x \cdot \overrightarrow{U} = 3\Omega N(T_e - T), \tag{0.3}$$

$$\frac{\partial}{\partial t}\mathcal{E}_e(T_e) + \nabla_x \cdot [\mathcal{E}_e(T_e)\overrightarrow{U}] + P_e\nabla_x \cdot \overrightarrow{U} = 3\Omega N(T - T_e), \tag{0.4}$$

where N, $\overrightarrow{U}$, T, and T_e are, respectively, the ionic density, the ionic macroscopic velocity, the ionic temperatures, and electronic temperatures. $\mathcal{E}_e(T_e) = \frac{3}{2}ZNT_e$ and $P_e = ZNT_e$ are the internal energy and the pressure of the electrons. $\Omega < 0$, defined with (4.1), is the collision frequency and is of the form $\Omega = N \cdot \Omega_0$, where Ω_0 depends continuously on N and $T_e^{-3/2}$ (see [2], [3], or [4]). For the numerical treatment of this macroscopic model, see [5] and [6]; for the physical analysis of this model and the link with a two fluid model, see, for example, [3].

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The kinetic model. Now we are concerned with the kinetic model. The ionic distribution $f = f(t, x, v)$ ($x \in \mathbf{R}^3$ and $\overrightarrow{v} \in \mathbf{R}^3$) and the electronic temperature $T_e(t, x)$ are solutions of

$$\frac{\partial}{\partial t} f + \overrightarrow{v} \cdot \nabla_x f - \frac{\nabla_x P_e}{Nm} \cdot \nabla_v f = B(f) + S(f), \tag{0.5}$$

$$\frac{\partial}{\partial t} \mathcal{E}_e(T_e) + \nabla_x \cdot [\mathcal{E}_e(T_e)\overrightarrow{U}] + P_e \nabla_x \cdot \overrightarrow{U} = -\frac{m}{2} \langle v^2 S(f) \rangle. \tag{0.6}$$

We have set $\langle \bullet \rangle = \int \bullet d\overrightarrow{v}$ and we define $N, \overrightarrow{U}$ by

$$N = \langle f \rangle, \quad N\overrightarrow{U} = \langle f \overrightarrow{v} \rangle.$$

The Fokker–Planck operator $S(f)$ is a model describing the collisions of the ions against the electrons and is defined with

$$S(f)(\overrightarrow{v}) = \Omega \nabla_v \cdot \left[(\overrightarrow{v} - \overrightarrow{U}) f + \frac{T_e}{m} \nabla_v f \right]. \tag{0.7}$$

$B(f)$ is the classical quadratic Landau operator (see [2], [3], and [4]). On the origin of the system (0.5) and (0.6), see [7] and the references therein; for a mathematical approach, see [12] and [13]. For some general features related to its numerical treatment, see [7] and [9] (in the stationary case).

The numerical solution of the overall system (0.5) and (0.6) can be done with a finite difference method in the phase space $(x, \overrightarrow{v})$ with a splitting in five stages:

(1) resolution of $\frac{\partial}{\partial t} f + \overrightarrow{v} \cdot \nabla_x f = 0$ with an upwind scheme with respect to the x variable;

(2) resolution of the ion/ion Fokker–Planck operator, i.e., we solve $\frac{\partial}{\partial t} f = B(f)$. For example, see [11], [14], [16], and [18] for a conservative and entropic scheme (and [17] in the isotropic case);

(3) resolution of $\frac{\partial}{\partial t} f = \frac{\nabla_x P_e}{Nm} \cdot \nabla_v f$ with an upwind scheme with respect to the $\overrightarrow{v}$ variable;

(4) resolution of the Fokker–Planck operator $S(f)$, i.e., we solve

$$\begin{cases} \dfrac{\partial}{\partial t} f = S(f), \\[2ex] \dfrac{\partial}{\partial t} \mathcal{E}_e = -\dfrac{m}{2} \langle v^2 S(f) \rangle; \end{cases} \tag{0.8}$$

(5) resolution of the remaining part of the electronic energy equation.

In this paper, we describe only the fourth stage, which is the most technical. Thus, we introduce the notion of *entropic average* which allows us to build a numerical scheme with very strong convergence and stability properties. To our knowledge, even the classical Chang–Cooper [8] numerical scheme does not realize all these properties (see also [9] and [10]).

Plan of the paper. In section 1, we give the main properties of the kinetic system: We check that there exists an entropy which is the sum of an ionic part equal to $\langle f \log f \rangle$ and an electronic part. This entropy is decreasing with time, and we check that if f is a Maxwellian function, the first three moments of (0.5) yield

the two-temperatures Euler equations. Section 2 is devoted to the introduction of the *entropic average* and to the analysis of a semidiscretization of the system (0.8) with respect to the velocity variable: the *entropic average* allows us to build a scheme whose discretized distribution f converges in large time t to the projection on the velocity grid of a Maxwellian distribution, properties which are not shown with other schemes as those described in [8] and [10].

In section 3, we describe the full discretization of the system (0.8): first, we build a positive and entropic explicit conservative scheme under a classical CFL criteria, and second, we build a semi-implicit conservative scheme which preserves the thermodynamical equilibrium. Finally, in section 4, we give numerical results which show that our scheme is more precise than the classical Chang–Cooper one (cf. [8]).

1. Preliminaries. Let us remark that even with a constant Ω, the operator $S(f)$ is not a linear operator with respect to f. Indeed, $\overrightarrow{U}$ depends on f and we can write

$$S(f)(\overrightarrow{v}) = \Omega_0 \nabla_v \cdot \left[\int (\overrightarrow{v} - \overrightarrow{v_*}) f(\overrightarrow{v_*}) f(\overrightarrow{v}) d\overrightarrow{v_*} + \int f(\overrightarrow{v_*}) d\overrightarrow{v_*} \frac{T_e}{m} \nabla_v f \right].$$

Let us define the Maxwellian

$$\mathbf{M}_{\overrightarrow{U},T}(\overrightarrow{v}) = \frac{N}{(2\pi T/m)^{3/2}} \exp\left[-\frac{m(\overrightarrow{v} - \overrightarrow{U})^2}{2T} \right].$$

We can write the operator $S(f)$ in the Landau form

$$S(f) = \Omega \frac{T_e}{m} \nabla_v \cdot \left[f \nabla_v \log\left(f/\mathbf{M}_{\overrightarrow{U},T_e} \right) \right]. \tag{1.1}$$

We do not emphasize in this paper the domain of the operator $S(f)$, but we assume in the following that f is a positive function belonging to $L^1[(1 + |\overrightarrow{v}|^2)dv]$ and that $|\overrightarrow{v}|^2 f(v) \to 0$ and $|\overrightarrow{v}|^2 \nabla_v f \to 0$ when $|\overrightarrow{v}| \to +\infty$.

For any f, we introduce an ionic temperature defined by

$$3NT = m\langle (\overrightarrow{v} - \overrightarrow{U})^2 f \rangle. \tag{1.2}$$

Using the properties

$$\int\int (\overrightarrow{v} - \overrightarrow{v_*}) f(\overrightarrow{v_*}) f(\overrightarrow{v}) d\overrightarrow{v_*} d\overrightarrow{v} = \overrightarrow{0},$$
$$m \int\int \overrightarrow{v}(\overrightarrow{v} - \overrightarrow{v_*}) f(\overrightarrow{v_*}) f(\overrightarrow{v}) d\overrightarrow{v_*} d\overrightarrow{v} + NT_e \int \overrightarrow{v} \nabla_v f d\overrightarrow{v} = 3N(T - T_e),$$

we can check that the operator $S(f)$ satisfies

$$\langle S(f) \rangle = 0, \quad \langle S(f)\overrightarrow{v} \rangle = \overrightarrow{0}, \quad \frac{m}{2}\langle S(f) v^2 \rangle = 3\Omega N(T_e - T). \tag{1.3}$$

Moreover, by using the Landau form and by assuming that $f = 0$ when $|\overrightarrow{v}| \to +\infty$, we easily see that

$$\left\langle \log\left(f/\mathbf{M}_{\overrightarrow{U},T_e} \right) S(f) \right\rangle = -\Omega \frac{T_e}{m} \int f \left[\nabla_v \log\left(f/\mathbf{M}_{\overrightarrow{U},T_e} \right) \right]^2 d\overrightarrow{v},$$

and we get the following lemma.

LEMMA 1.1. *For all $f > 0$ and $T_e > 0$, we have*

$$\left\langle S(f) \log \left(f/\mathbf{M}_{\overrightarrow{U},T_e} \right) \right\rangle \leq 0. \tag{1.4}$$

Moreover, $\langle S(f) \log(f/\mathbf{M}_{\overrightarrow{U},T_e}) \rangle = 0$ *if and only if* $f = \mathbf{M}_{\overrightarrow{U},T_e}$.

We know that the operator $B(f)$ conserves the mass, the momentum, and the energy and we have $\langle B(f) \log f \rangle \leq 0$. Thus

$$\left\langle B(f) \log \left(f/\mathbf{M}_{\overrightarrow{U},T_e} \right) \right\rangle \leq 0. \tag{1.5}$$

Lemma 1.1 with (1.5) gives the following proposition.

PROPOSITION 1.2. *Let f and T_e be solutions of* (0.5) *and* (0.6) *with T_e smooth enough with respect to the variable x. Then we have the relation of the decay of the entropy*

$$\frac{\partial}{\partial t} \left(\langle f \log f \rangle + \mathcal{H}_e \right) + \nabla_x \cdot \left(\langle \overrightarrow{v} f \log f \rangle + \overrightarrow{U} \mathcal{H}_e \right) \leq 0, \quad \textit{where} \quad \mathcal{H}_e = ZN \log(N T_e^{-3/2}). \tag{1.6}$$

Proof of Proposition 1.2. Let us denote $\overline{\overline{P_i}} = m \langle f (\overrightarrow{v} - \overrightarrow{U}) \otimes (\overrightarrow{v} - \overrightarrow{U}) \rangle$. By taking the two first moments of the kinetic equation (0.5), a classical calculus yields

$$\frac{\partial}{\partial t} N + \nabla_x \cdot (N \overrightarrow{U}) = 0,$$

$$\frac{\partial}{\partial t} (m N \overrightarrow{U}) + \nabla_x \cdot (m N \overrightarrow{U} \otimes \overrightarrow{U}) + \nabla_x (Z N T_e) + \nabla_x \cdot \overline{\overline{P_i}} = \overrightarrow{0}. \tag{1.7}$$

Now, using the classical relation

$$\frac{\partial}{\partial t} (N w) + \nabla_x \cdot (N w \overrightarrow{U}) = N \left(\frac{\partial}{\partial t} w + \overrightarrow{U} \cdot \nabla_x w \right), \tag{1.8}$$

which is valid for any w, we get

$$N \left(\frac{\partial}{\partial t} (N^{-1}) + \overrightarrow{U} \cdot \nabla_x (N^{-1}) \right) - \nabla_x \cdot \overrightarrow{U} = 0, \tag{1.9}$$

$$m N \left(\frac{\partial}{\partial t} \overrightarrow{U} + \overrightarrow{U} \cdot \nabla_x \overrightarrow{U} \right) + \nabla_x (Z N T_e) + \nabla_x \cdot \overline{\overline{P_i}} = \overrightarrow{0}.$$

If we multiply the kinetic equation (0.5) with $m v^2/2$, we get

$$m \frac{\partial}{\partial t} \left\langle \frac{v^2}{2} f \right\rangle + m \nabla_x \cdot \left\langle \overrightarrow{v} \frac{v^2}{2} f \right\rangle + \overrightarrow{U} \cdot \nabla_x (Z N T_e) = 3 \Omega N (T_e - T). \tag{1.10}$$

(i) *The electronic entropy.* According to (0.4) and (1.8), the electronic energy balance equation may be written as

$$\frac{3}{2} \left(\frac{\partial T_e}{\partial t} + \overrightarrow{U} \cdot \nabla_x T_e \right) + T_e \nabla_x \cdot \overrightarrow{U} = 3 \frac{\Omega}{Z} (T - T_e).$$

Thus

$$\frac{\partial}{\partial t}\log(NT_e^{-3/2}) + \overrightarrow{U}\cdot\nabla_x \log(NT_e^{-3/2}) = 3\frac{\Omega}{Z}\frac{(T_e - T)}{T_e};$$

that is to say

$$\frac{\partial \mathcal{H}_e}{\partial t} + \nabla_x\cdot(\overrightarrow{U}\mathcal{H}_e) = 3\Omega N\frac{(T_e - T)}{T_e}. \tag{1.11}$$

(ii) *The ionic entropy.* According to Lemma 1.1, the inequality (1.5), the relation $\langle \partial_t f + \overrightarrow{v}\cdot\nabla_x f\rangle = 0$, and the relation $\langle\nabla_v f\rangle = \overrightarrow{0}$, we see that

$$\left\langle\left(\log f + \frac{m}{2}\frac{(\overrightarrow{v}-\overrightarrow{U})^2}{T_e}\right)\frac{\partial}{\partial t}f\right\rangle + \left\langle\left(\log f + \frac{m}{2}\frac{(\overrightarrow{v}-\overrightarrow{U})^2}{T_e}\right)\nabla_x\cdot(f\overrightarrow{v})\right\rangle \le 0.$$

Since $\langle f\frac{\partial}{\partial t}\log f\rangle + \langle f\overrightarrow{v}\cdot\nabla_x\log f\rangle = 0$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\langle f\log f\rangle + \nabla_x\cdot\langle\overrightarrow{v}f\log f\rangle \le \frac{m}{T_e}\Big[-\Big\langle\frac{\partial}{\partial t}\frac{v^2}{2}f\Big\rangle + \Big\langle\overrightarrow{v}\cdot\overrightarrow{U}\frac{\partial}{\partial t}f\Big\rangle - \nabla_x\cdot\Big\langle\overrightarrow{v}\frac{v^2}{2}f\Big\rangle \\ + \langle(\overrightarrow{v}\cdot\overrightarrow{U})\nabla_x\cdot(f\overrightarrow{v})\rangle\Big].\end{aligned}$$

On the other hand, we can notice that

$$\begin{aligned} m\left(\left\langle\overrightarrow{v}\cdot\overrightarrow{U}\frac{\partial}{\partial t}f\right\rangle + \langle(\overrightarrow{v}\cdot\overrightarrow{U})\nabla_x\cdot(f\overrightarrow{v})\rangle\right) &= m\left[\overrightarrow{U}\cdot\partial_t(N\overrightarrow{U}) + \overrightarrow{U}\cdot\langle\nabla_x\cdot(f\overrightarrow{v}\otimes\overrightarrow{v})\rangle\right] \\ &= \overrightarrow{U}\cdot\left[\partial_t(mN\overrightarrow{U}) + \nabla_x\cdot(mN\overrightarrow{U}\otimes\overrightarrow{U})\right] \\ &\quad + \overrightarrow{U}\cdot\nabla_x\overline{\overline{P_i}}.\end{aligned}$$

However, according to (1.7), this last expression is equal to

$$-\overrightarrow{U}\cdot\nabla_x\overline{\overline{P_i}} - \overrightarrow{U}\cdot\nabla_x(ZNT_e) + \overrightarrow{U}\cdot\nabla_x\overline{\overline{P_i}} = -\overrightarrow{U}\cdot\nabla_x(ZNT_e).$$

By gathering the previous relations and (1.10), we get

$$\frac{\partial}{\partial t}\langle f\log f\rangle + \nabla_x\cdot\langle\overrightarrow{v}f\log f\rangle \le 3\Omega N\frac{(T - T_e)}{T_e}$$

which, by adding (1.11), gives the result. □

Remark. If the ionic distribution function is Maxwellian, we have $\overline{\overline{P_i}} = \overline{\overline{1}}NT$ and, according to (1.10), we obtain

$$\left(\frac{\partial}{\partial t} + \nabla_x\cdot(\overrightarrow{U}\cdot)\right)\left(\frac{3}{2}NT + N\frac{m}{2}|\overrightarrow{U}|^2\right) + \overrightarrow{U}\cdot\nabla_x(ZNT_e) + \nabla_x\cdot(\overrightarrow{U}NT) = 3\Omega N(T_e - T).$$

Thus, according to (1.7), we see that (0.3) is satisfied. Then, N, U, T, and T_e satisfy the fluid system (0.1), (0.2), (0.3), and (0.4).

2. Semidiscretized scheme for the ion/electron Fokker–Planck operator. For the sake of simplicity, we shall consider only the monodimensional Cartesian case; that is, f depends only of $t \in \mathbf{R}^+$ and $v \in \mathbf{R}$. Then, $\mathcal{E}_e(T_e) = \frac{Z}{2} N T_e$ and $3NT$ has to be replaced by NT in (1.2) and the system (0.8) becomes

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} f = S(f), \\ \\ \frac{Z}{2} \frac{\partial}{\partial t} (N T_e) = \Omega N (T - T_e) \end{array} \right. \tag{2.1}$$

with

$$S(f) = \Omega \partial_v \left[(v - U) f + \frac{T_e}{m} \partial_v f \right].$$

We will study discretization in velocity only (semidiscretized model). We use a discretization of $\mathbf{R}$ by a finite difference grid $\{v_j\}$ ($j \in \{1, \dots, j_{max}\}$) and $\Delta v = v_{j+1} - v_j$ is constant. Let us note $f_j(t)$, the evaluation of $f(t)$ at the point v_j. From now on, we set

$$\langle \Psi \rangle = \Delta v \sum_j \Psi_j.$$

Let us recall that, in the continuous case, the collision operator $S(f)$ verifies the conservation properties (1.3), which implies in the homogeneous case that $\partial_t N = 0$ and $\partial_t U = 0$; these properties will have to be verified in the discretized case (which will be verified; see Proposition 2.2).

2.1. Definition of the semidiscretized scheme and preliminary results. Boundary conditions in the general continuous case for a compact velocity domain. Since we will consider compact velocity domain $\mathcal{V}$ for the numerical applications, we briefly study the boundary conditions that we have to impose in the general continuous case to verify again the conservation properties (1.3) when the velocity domain is compact. The first consequence is that we cannot say that f and $\nabla_v f$ are equal to zero on the frontier $\delta\mathcal{V}$ of the compact velocity domain and, then, we must give new boundary conditions even for the general continuous system; for this one, in order to ensure the mass conservation, the best boundary conditions are the Robin ones

$$(\overrightarrow{v} - \overrightarrow{U}) f + \frac{T_e}{m} \nabla_v f = \overrightarrow{0} \quad \text{when} \quad \overrightarrow{v} \in \delta\mathcal{V}. \tag{2.2}$$

And, to ensure the momentum conservation, we have to define the macroscopic velocity $\overrightarrow{U}$ with

$$\overrightarrow{U} = \frac{\int_{\mathcal{V}} \overrightarrow{v} f(v) dv}{\int_{\mathcal{V}} f(v) dv} + \delta \overrightarrow{u}, \tag{2.3}$$

where

$$\delta \overrightarrow{u} = \frac{T_e}{m \int_{\mathcal{V}} f(v) dv} \int_{\delta\mathcal{V}} f(v) d\overrightarrow{\mathcal{S}}, \tag{2.4}$$

$d\overrightarrow{\mathcal{S}}$ being the measure of $\delta\mathcal{V}$. And, due to the corrective term $\delta\overrightarrow{u}$, it is easy to verify that

$$\int_{\mathcal{V}} m \frac{v^2}{2} S(f) dv = d\Omega \cdot (T_e - T) \cdot \int_{\mathcal{V}} f(v) dv \tag{2.5}$$

(d is the dimension of the velocity space $\mathcal{V}$) with

$$\left\{\begin{array}{l} T = \frac{m}{d\cdot\int_{\mathcal{V}} f(v)dv} \displaystyle\int_{\mathcal{V}} (\overrightarrow{v} - \overrightarrow{U})^2 f(v)dv + \delta t, \\ \delta t = \frac{T_e}{d\cdot\int_{\mathcal{V}} f(v)dv} \displaystyle\int_{\delta\mathcal{V}} (v - U) f(v) d\overrightarrow{\mathcal{S}}. \end{array}\right. \tag{2.6}$$

We will use this remark at the discretized level when $j = 1$ and $j = j_{max}$. Indeed, we define the following numerical scheme with the following discrete boundary conditions.

Statement of the semidiscrete scheme. For initial conditions f^0 and T_e^0, we consider the following scheme:

$$\left\{\begin{array}{ll} \partial_t f_j = S(f)_j & \forall j \in \{1, \dots, j_{max}\} : f_j(0) = f_j^0, \\ \frac{Z}{2}\partial_t(NT_e) = \Omega(\widetilde{NT} - NT_e), & T_e(0) = T_e^0, \end{array}\right. \tag{2.7}$$

with

$$S(f)_j = \frac{\Omega}{\Delta v}\left[(v_{j+1/2} - \widetilde{U})\widetilde{f}_{j+1/2} - (v_{j-1/2} - \widetilde{U})\widetilde{f}_{j-1/2}\right] + \frac{\Omega T_e}{m\Delta v^2}(a_j f_{j+1} - b_j f_j + c_j f_{j-1}) \tag{2.8}$$

and

$$N = \langle f \rangle,$$

$$\left\{\begin{array}{l} \widetilde{N} = \displaystyle\sum_j \widetilde{f}_{j+1/2}\Delta v, \\ \widetilde{U} = \displaystyle\sum_j v_{j+1/2}\widetilde{f}_{j+1/2}\Delta v/\widetilde{N} + \delta\widetilde{u}, \\ \widetilde{NT} = \displaystyle\sum_j m(v_{j+1/2} - \widetilde{U})^2\widetilde{f}_{j+1/2}\Delta v + \widetilde{N}\cdot\delta\widetilde{t}. \end{array}\right. \tag{2.9}$$

$\widetilde{f}_{j+1/2}(t)$ is an appropriate average of $f_j(t)$ and $f_{j+1}(t)$: we will see that we will use the *entropic average* (see Definition 2.1) to define this average.

$\delta\widetilde{u}$ and $\delta\widetilde{t}$ are corrective terms necessary for the conservation of momentum and energy, knowing that the distribution f is not always equal to zero on the boundary of $[v_1, v_{j_{\max}}]$. These terms are defined in the following way:

$$\left\{\begin{array}{l} \delta\widetilde{u} = \frac{T_e}{m\widetilde{N}}\left(f_{j_{\max}} - f_1\right), \\ \delta\widetilde{t} = \frac{T_e}{\widetilde{N}}\left[f_{j_{\max}}\left(v_{j_{\max}+1/2} - \widetilde{U}\right) + f_1\left(\widetilde{U} - v_{1/2}\right)\right]. \end{array}\right. \tag{2.10}$$

Let us remark that (2.9) and (2.10) are the discrete versions of the relations (2.3), (2.4), and (2.6) (see the proof of Proposition 2.2 for a justification of these formulas).

From a practical point of view, we can say that when $|v_1|$ and $|v_{j_{max}}|$ will be large enough, these corrective terms will be very small since the distributions $\{f_j\}$ converge to the projection of a Maxwellian distribution on the velocity mesh $\{v_j\}$

when t goes to infinity; see Theorem 2.8. Then, it would be possible to forget them for the numerical applications.

Discrete boundary conditions. To take into account the Robin's boundary conditions (2.2) at the boundary of the velocity domain $\mathcal{V} = [v_1, v_{j_{max}}]$, we set

$$\begin{cases} a_j = 1 \text{ if } j \neq j_{\max}, \\ b_j = 2 \text{ if } j \in \{2, \dots, j_{\max} - 1\}, \\ c_j = 1 \text{ if } j \neq 1, \\ b_1 = b_{j_{\max}} = 1 \text{ and } a_{j_{\max}} = c_1 = 0 \end{cases} \tag{2.11}$$

and

$$\widetilde{f}_{1/2} = \widetilde{f}_{j_{\max}+1/2} \equiv 0. \tag{2.12}$$

Now, we introduce the *entropic average*.

DEFINITION 2.1. *The entropic average $\widetilde{f}$ of two strictly positive quantities x and y is defined by*

$$\widetilde{f} = \begin{cases} \frac{x-y}{\log x - \log y} & \text{if } x \neq y, \\ x & \text{otherwise.} \end{cases}$$

By continuity, we extend this definition by setting $\widetilde{f} = 0$ if $x = 0$ or $y = 0$.

Notice that for any smooth and positive function f, we have

$$f(v + \Delta v/2) = \frac{f(v + \Delta v) - f(v)}{\log f(v + \Delta v) - \log f(v)} + O(\Delta v^2),$$

which makes the discretized operator (2.8) of second order in velocity space.

Preliminary results.

PROPOSITION 2.2. *The conservation laws*

$$\begin{cases} \langle S(f) \rangle = 0, \\ \langle vS(f) \rangle = 0, \\ m\langle \frac{v^2}{2} S(f) \rangle = \Omega(NT_e - \widetilde{NT}) \end{cases}$$

are verified. Thus, $N \equiv \langle f \rangle$, $NU \equiv \langle vf \rangle$, and $m\langle \frac{(v-U)^2}{2} f \rangle + \frac{Z}{2} NT_e$ do not depend on t.

Proof of Proposition 2.2. The first relation comes from

$$\begin{aligned} \langle S(f) \rangle = {} & \frac{\Omega}{\Delta v} \left[(v_{j_{\max}+1/2} - \widetilde{U}) \widetilde{f}_{j_{\max}+1/2} - (v_{1/2} - \widetilde{U}) \widetilde{f}_{1/2} \right] \\ & + \frac{\Omega T_e}{m \Delta v^2} \left[\sum_{j=2}^{j_{\max}} f_j - \sum_{j=1}^{j_{\max}-1} f_j - \sum_{j=2}^{j_{\max}} f_j + \sum_{j=1}^{j_{\max}-1} f_j \right] = 0. \end{aligned}$$

We also have

$$\begin{aligned} \langle vS(f) \rangle &= -\Omega \sum_{j=1}^{j_{\max}} (v_{j+1/2} - \widetilde{U}) \widetilde{f}_{j+1/2} \Delta v + \frac{\Omega T_e}{m} (f_1 - f_{j_{\max}}) \\ &= \Omega \delta \widetilde{u} \sum_{j=1}^{j_{\max}} \widetilde{f}_{j+1/2} \Delta v + \frac{\Omega T_e}{m} (f_1 - f_{j_{\max}}) = 0. \end{aligned}$$

For the variation of the ionic energy, we obtain

$$\begin{aligned}
\Omega^{-1} m \left\langle \frac{v^2}{2} S(f) \right\rangle &= -\sum_{j=1}^{j_{\max}} m(v_{j+1/2} - \widetilde{U}) v_{j+1/2} \widetilde{f}_{j+1/2} \Delta v \\
&\quad - T_e \left(\sum_{j=2}^{j_{\max}} f_j v_{j-1/2} - \sum_{j=1}^{j_{\max}-1} f_j v_{j+1/2} \right) \\
&= -\sum_{j=1}^{j_{\max}} m(v_{j+1/2} - \widetilde{U}) v_{j+1/2} \widetilde{f}_{j+1/2} \Delta v \\
&\quad - T_e \left(f_{j_{\max}} v_{j_{\max}-1/2} - f_1 v_{3/2} - \sum_{j=2}^{j_{\max}-1} f_j \Delta v \right) \\
&= -\sum_{j=1}^{j_{\max}} m(v_{j+1/2} - \widetilde{U})^2 \widetilde{f}_{j+1/2} \Delta v + m\widetilde{N}\widetilde{U}\delta\widetilde{u} \\
&\quad - T_e \left(f_{j_{\max}} v_{j_{\max}+1/2} - f_1 v_{1/2} - N \right) \\
&= (NT_e - \widetilde{NT}). \qquad \square
\end{aligned}$$

Now, we introduce the numerical entropy

$$H(f, T_e) = \langle f \log f \rangle - \frac{ZN}{2} \log T_e$$

and the Maxwellian

$$\mathbf{M}_{\widetilde{U},T_e}(v,t) = \frac{N}{\sqrt{2\pi T_e/m}} \exp\left[-\frac{m(v-\widetilde{U})^2}{2T_e} \right]$$

associated with (f, T_e). Using the fact that

$$\langle v^2 S(f) \rangle = \langle (v - \widetilde{U})^2 S(f) \rangle, \tag{2.13}$$

we can show that

$$\frac{\partial}{\partial t} H(f, T_e) = \langle S(f) \log f \rangle - \frac{\Omega(\widetilde{NT} - NT_e)}{T_e} = \left\langle S(f) \log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}} \right) \right\rangle. \tag{2.14}$$

This is still true by defining $\widetilde{f}_{j+1/2}(t)$ in another way, but the entropic average allows us to have the following results.

LEMMA 2.3. *For all strictly positive $f_j(t)$ and $T_e(t)$, we have*

$$\begin{aligned}
S(f) = \frac{\Omega T_e}{m\Delta v^2} \Bigg\{ &\widetilde{f}_{j+1/2} \left[\log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}} \right)_{j+1} - \log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}} \right)_j \right] \\
&- \widetilde{f}_{j-1/2} \left[\log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}} \right)_j - \log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}} \right)_{j-1} \right] \Bigg\}
\end{aligned} \tag{2.15}$$

and

$$(2.16)\ \frac{\partial}{\partial t} H(f,T_e) = -\frac{\Omega T_e}{m\Delta v^2} \sum \widetilde{f}_{j+1/2} \left[\log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}}\right)_{j+1} - \log\left(\frac{f}{\mathbf{M}_{\widetilde{U},T_e}}\right)_j \right]^2 \leq 0.$$

This lemma shows that there is equivalence between the convection-diffusion form (0.7) and the Landau form (1.1) of the discretized Fokker–Planck collision operator and that the semidiscretized scheme is entropic: these strong properties, due to the definition of the *entropic average*, are essential to obtain the convergence results in the next paragraph.

Proof of Lemma 2.3. The first point comes from

$$\log(\mathbf{M}_{\widetilde{U},T_e})_{j+1} - \log(\mathbf{M}_{\widetilde{U},T_e})_j = -\frac{m\Delta v}{T_e}(v_{j+1/2} - \widetilde{U}). \tag{2.17}$$

The second one comes from the first point and from (2.14). ∎

And now we can show the following result.

Proposition 2.4. *For all strictly positive initial conditions, the semidiscretized scheme defined by* (2.7) *and* (2.8) *has a global positive solution and*

$$\inf_{t\in[0,+\infty[} T_e(t) > 0.$$

The proof of Proposition 2.4 is in the appendix.

Corollary 2.5. *There exists a constant H^∞ such that*

$$\lim_{t\to+\infty} H(f,T_e) = H^\infty.$$

Proof of Corollary 2.5. A direct consequence of Proposition 2.4 is that $H(f,T_e)$ is well defined on $[0,+\infty[$. By applying Lemma 2.3, we see that $H(f,T_e)$ is a decreasing function. Since the scheme conserves the energy, $T_e(t)$ is bounded from above, which implies that $H(f,T_e)$ is bounded from below on $[0,+\infty[$ since $x \mapsto x\log x$ is bounded from below on $[0,+\infty[$. Therefore, $H(f,T_e)$ has a limit $H^\infty > -\infty$ when t goes to infinity. ∎

2.2. Convergence of the semidiscretized scheme toward an unique equilibrium. Now we define the numerical equilibrium state f_j^∞ which is a projection of a Maxwelian distribution on the discretized velocity grid $\{v_j\}$.

Definition 2.6. *The numerical equilibrium state f_j^∞ is defined with*

$$f_j^\infty = N \cdot \frac{\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}(v_j)}{\langle \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}\rangle}, \tag{2.18}$$

where

$$\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}(v) = \frac{N}{\sqrt{2\pi T_e^\infty/m}} \exp\left[-\frac{m(v-\widetilde{U}^\infty)^2}{2T_e^\infty}\right], \tag{2.19}$$

knowing that $(\widetilde{U}^\infty, T_e^\infty)$ is a solution of the nonlinear system

$$(2.20)\ \begin{cases} \frac{N}{\langle \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}\rangle} \cdot \langle v\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}\rangle = \langle vf^0\rangle, \\[2ex] \frac{N}{\langle \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}\rangle} \cdot \langle (v-U^0)^2\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}\rangle + \frac{ZN}{m}T_e^\infty = \langle (v-U^0)^2 f^0\rangle + \frac{ZN}{m}T_e^0. \end{cases}$$

Let us notice that $\langle \mathbf{M}_{U,T}\rangle = N\sum_j \exp[-\frac{m(v-U)^2}{2T}]\frac{\Delta v}{\sqrt{2\pi T/m}}$ is not exactly equal to N according to the discretization errors.

We can now state the following result.

Proposition 2.7. *For all strictly positive initial conditions,*

(i) *the semidiscretized scheme defined by* (2.7) *and* (2.8) *ensures that there exists a subsequence* (t_k) *such that the functions* $f_j(t_k)$ *converge to the equilibrium state* f_j^∞ *given by* (2.18)*;*

(ii) *any solution* $(\widetilde{U}^\infty, T_e^\infty)$ *of* (2.20) *satisfies*

$$\forall t : H(f^\infty, T_e^\infty) \le H(f, T_e);$$

(iii) *the system* (2.20) *admits a unique solution.*

The proof of Proposition 2.7 is in the appendix.

Now we write the main result of this section.

Theorem 2.8. *For all strictly positive initial conditions, the semidiscretized scheme defined by* (2.7) *and* (2.8) *ensures that*

(i)

$$\lim_{t\to+\infty} f_j(t) = f_j^\infty,$$

(ii)

$$\lim_{t\to+\infty} T_e(t) = T_e^\infty,$$

where f_j^∞ *and* T_e^∞ *are given by the unique equilibrium state defined with* (2.18), (2.19), *and* (2.20).

Let us remark that

$$\widetilde{U}^\infty \simeq U^0 = \frac{\langle v f^0\rangle}{N}$$

and that

$$T_e^\infty \simeq \frac{T^0 + ZT_e^0}{1+Z},$$

where $NT^0 = \langle \left(v - U^0\right)^2 f^0\rangle$.

Proof of Theorem 2.8. We know that $H(f, T_e)$ has a limit (see Corollary 2.5) and that $H(f, T_e)$ is bounded from below by $H(f^\infty, T_e^\infty)$ (see point (ii) of Proposition 2.7). Then we can write

$$\lim_{t\to+\infty} H(f, T_e) - H(f^\infty, T_e^\infty) = a \ge 0.$$

On the other hand, using point (i) of Proposition 2.7, we can say that there exists a sequence (t_k) such that

$$\lim_{t_k\to+\infty} f_j(t_k) = f_j^\infty.$$

Then, for the sequence (t_k), we also have

$$\lim_{t_k\to+\infty} H(f, T_e)(t_k) - H(f^\infty, T_e^\infty) = 0.$$

Then, $a = 0$; that is,

$$\lim_{t\to+\infty} H(f,T_e)(t) - H(f^\infty, T_e^\infty) = 0. \tag{2.21}$$

However, the Csiszar–Kullback inequality (cf. [21] and [22]) allows us to write that

$$\left\| f_j - f_j^\infty \right\|_{l_1}^2 \leq 2 \sum_j f_j \log\left(\frac{f_j}{f_j^\infty}\right) \Delta v$$

and

$$\left\| \mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \right\|_{L_1}^2 \leq 2 \int \mathbf{M}_{\widetilde{U}^\infty, T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty, T_e}}{\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}}\right) dv.$$

Then, by applying Lemma A.1 (see the appendix) with the limit (2.21), we obtain

$$\lim_{t\to+\infty} \left\| f_j - f_j^\infty \right\|_{l_1} = 0$$

and

$$\lim_{t\to+\infty} \left\| \mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \right\|_{L_1} = 0,$$

this last limit showing that $\lim_{t\to+\infty} T_e(t) = T_e^\infty$. ∎

3. Fully discretized scheme. Let us denote by a superscript n the values of the various variables at the time t^n and let us define the time step $\Delta t = t^{n+1} - t^n$.

3.1. Time explicit scheme. We define the following explicit scheme:

$$\begin{cases} \frac{1}{\Delta t}(f_j^{n+1} - f_j^n) = S(f^n)_j, \\ \frac{1}{\Delta t}\left[\frac{Z}{2}\,(NT_e)^{n+1} - \frac{Z}{2}\,(NT_e)^n\right] = \Omega^n(\widetilde{NT}^n - N^n T_e^n) \end{cases} \tag{3.1}$$

with

$$\begin{aligned} S(f^n)_j = {} & \frac{\Omega^n}{\Delta v}\left[(v_{j+1/2} - \widetilde{U}^n)\widetilde{f}^n_{j+1/2} - (v_{j-1/2} - \widetilde{U}^n)\widetilde{f}^n_{j-1/2}\right] \\ & + \frac{\Omega^n T_e^n}{m\Delta v^2}(a_j f_{j+1}^n - b_j f_j^n + c_j f_{j-1}^n). \end{aligned} \tag{3.2}$$

$\widetilde{f}^n_{j+1/2}$ is the entropic average of f_j^n and f_{j+1}^n (see Definition 2.1); a_j, b_j, and c_j are given by (2.11); and we set $\widetilde{f}^n_{1/2} = \widetilde{f}^n_{j_{\max}+1/2} \equiv 0$. $\widetilde{U}^n$ and $\widetilde{NT}^n$ are defined by (2.9); and Ω^n is evaluated with the values known at time t^n. We obtain the following conservation relations.

Proposition 3.1. *The conservation laws*

$$\begin{cases} \langle f^{n+1}\rangle = \langle f^n\rangle, \\ \langle v f^{n+1}\rangle = \langle v f^n\rangle, \\ m\langle \frac{v^2}{2} f^{n+1}\rangle - m\langle \frac{v^2}{2} f^n\rangle = \Delta t\,\Omega^n(N^n T_e^n - \widetilde{NT}^n) \end{cases}$$

are verified. Thus, $N^n \equiv \langle f^n \rangle \equiv N$, $N^n U^n \equiv \langle v f^n \rangle \equiv NU$, *and* $m\langle \frac{(v-U^n)^2}{2} f^n \rangle + \frac{Z}{2} N^n T_e^n \equiv \frac{N}{2}(T^0 + ZT_e^0)$ *do not depend on* n.

That is, the scheme conserves the mass, the momentum, and the energy, which is consistent with (1.3). Let us note that the equation for the electronic energy can be written in the following way:

$$T_e^{n+1} = T_e^n \left(1 - \Delta t \frac{2\Omega^n}{Z}\right) + 2\Delta t \Omega^n \frac{\widetilde{NT}^n}{ZN}. \tag{3.3}$$

Positivity and conservation of the equilibrium state. Now we show that the time explicit scheme defined by (3.1) and (3.2) is positive under a *CFL* criteria and verifies a discrete version of the H-theorem. We set

$$\Delta t_1^n = \frac{m}{4\Omega^n T_e^n} \cdot \frac{\Delta v^2}{\mathfrak{M}^n} \qquad \text{and} \qquad \Delta t_2^n = \frac{Z}{2\Omega^n} \tag{3.4}$$

with

$$\mathfrak{M}^n = \max_j \left[\frac{\mathbf{M}_{\widetilde{U}^n, T_e^n, j\pm 1}}{\mathbf{M}_{\widetilde{U}^n, T_e^n, j}} \right]$$

(where j and $j \pm 1 \in \{1, \dots, j_{\max}\}$). Let us recall that

$$\mathbf{M}_{\widetilde{U}^n, T_e^n, j} = \frac{N}{\sqrt{2\pi T_e^n / m}} \exp\left[-\frac{m(v_j - \widetilde{U}^n)^2}{2T_e^n} \right],$$

that f^∞ is defined by (2.18), and that $(\widetilde{U}^\infty, T_e^\infty)$ is the unique solution of the system (2.20).

We have the following result.

PROPOSITION 3.2. *For all strictly positive initial conditions, the explicit scheme defined by* (3.1) *and* (3.2) *preserves the positivity of* f_j^{n+1} *and* T_e^{n+1} *and verifies a discrete version of the H-theorem*

$$\langle S(f^n) \log(f^n / \mathbf{M}_{\widetilde{U}^n, T_e^n}) \rangle \leq 0$$

under the CFL criteria

$$\Delta t < \min(\Delta t_1^n, \Delta t_2^n). \tag{3.5}$$

Moreover, we have

$$f^n = f^\infty \quad \textit{and} \quad T_e^n = T_e^\infty > 0 \quad \iff \quad f^{n+1} = f^n \quad \textit{and} \quad \forall j : f_j^n > 0,\ T_e^n > 0.$$

The proof of Proposition 3.2 is in the appendix.

The decay of the entropy. Now we show that the scheme is entropic, that (T_e^n) is bounded from below, and that the time step Δt does not vanish in finite time under a very weak hypothesis. We set

$$\Delta t_3^n = \Delta t_1^n \cdot \frac{h_{\min}^n}{h_{\max}^n} \cdot \frac{1}{1+\alpha^n} \quad \text{and} \quad \Delta t_4^n = \frac{1}{2}\Delta t_2^n,$$

$$\begin{cases} h_{\max}^n = \max_k \left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}} \right)_k, \\ h_{\min}^n = \min_k \left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}} \right)_k, \\ \alpha^n = \frac{1}{Z} \cdot \frac{\max_k (v_k - \widetilde{U}^n)^4}{(T_e^n/m)^2}, \end{cases}$$

and

$$H^n = \langle f^n \log f^n \rangle - \frac{ZN}{2} \log T_e^n.$$

Proposition 3.3. *For all strictly positive initial conditions, the scheme defined by* (3.1) *and* (3.2) *verifies the inequality*

$$H(f^\infty, T_e^\infty) \le H^{n+1} \le H^n$$

if we have

$$\Delta t < \min(\Delta t_3^n, \Delta t_4^n). \tag{3.6}$$

The proof of Proposition 3.3 is in the appendix.

Corollary 3.4. *For all strictly positive initial conditions, and under the CFL condition* (3.6)*, we have*

$$\inf_n T_e^n > 0,$$

and there exists a constant H^∞ such that

$$\lim_{n \to +\infty} H^n = H^\infty.$$

Since $\inf_n T_e^n > 0$ (see Corollary 3.4), we also have $\inf_n \Delta t_4^n > 0$. However, we do not have $\inf_n \Delta t_3^n > 0$ because this property is related to the property $\inf_{j,n}(f_j^n) > 0$ which seems to be difficult to obtain. (This lower bound would prove that the explicit scheme converges to the equilibrium state f_j^∞ given by (2.18).) However, we have to check on the numerical experiments that the coefficient Δt_3^n does not vanish.

3.2. Time semi-implicit scheme. Despite its good behavior, the explicit scheme is expensive since the *CFL* condition is in Δv^2. This leads us to implicate the diffusive term. Then, we define the semi-implicit scheme as

$$\begin{cases} \frac{1}{\Delta t}(f_j^{n+1} - f_j^n) = S(f^n, f^{n+1})_j, \\ \\ \frac{1}{\Delta t}\left[\frac{1}{2}(N_e T_e)^{n+1} - \frac{1}{2}(N_e T_e)^n\right] = \Omega^n(\widetilde{NT}^{n+1/2} - NT_e^n) \end{cases} \tag{3.7}$$

with

$$\begin{aligned} S(f^n, f^{n+1})_j = {}& \frac{\Omega^n}{\Delta v}\left[(v_{j+1/2} - \widetilde{U}^{n+1/2})\widetilde{f}_{j+1/2}^n - (v_{j-1/2} - \widetilde{U}^{n+1/2})\widetilde{f}_{j-1/2}^n\right] \\ & + \frac{\Omega^n T_e^n}{m\Delta v^2}(a_j f_{j+1}^{n+1} - b_j f_j^{n+1} + c_j f_{j-1}^{n+1}). \end{aligned} \tag{3.8}$$

As in the explicit case, $\widetilde{f}^n_{j+1/2}$ is the entropic average of f^n_j and f^n_{j+1}; a_j, b_j, and c_j are given by (2.11) and we set $\widetilde{f}^n_{1/2} = \widetilde{f}^n_{j_{\max}+1/2} \equiv 0$. In (3.7) and (3.8), $\widetilde{U}^{n+1/2}$ and $\widetilde{NT}^{n+1/2}$ (defined with (2.9)) are semi-implicit only through the corrective terms $\delta\widetilde{u}^{n+1/2}$ and $\delta\widetilde{t}^{n+1/2}$ defined with

$$\begin{cases} \delta\widetilde{u}^{n+1/2} = \frac{T_e^n}{m\widetilde{N}^n}\left(f^{n+1}_{j_{\max}} - f^{n+1}_1\right), \\ \delta\widetilde{t}^{n+1/2} = \frac{T_e^n}{\widetilde{N}^n}\left[f^{n+1}_{j_{\max}}\left(v_{j_{\max}+1/2} - \widetilde{U}^{n+1/2}\right) + f^{n+1}_1\left(\widetilde{U}^{n+1/2} - v_{1/2}\right)\right]. \end{cases}$$

However, for and only for the numerical applications, we will neglect the corrective terms $\delta\widetilde{u}^{n+1/2}$ and $\delta\widetilde{t}^{n+1/2}$, and then $\widetilde{U}^{n+1/2}$ and $\widetilde{NT}^{n+1/2}$ will be completely explicit.

We obtain the following conservation laws.

PROPOSITION 3.5. *The conservation laws*

$$\begin{cases} \langle f^{n+1}\rangle = \langle f^n\rangle, \\ \langle vf^{n+1}\rangle = \langle vf^n\rangle, \\ m\langle \frac{v^2}{2}f^{n+1}\rangle - m\langle \frac{v^2}{2}f^n\rangle = \Delta t\Omega^n(N^nT_e^n - \widetilde{NT}^{n+1/2}) \end{cases} \tag{3.9}$$

are verified. Thus, $N^n \equiv \langle f^n\rangle \equiv N$, $N^nU^n \equiv \langle vf^n\rangle \equiv NU$, *and* $m\langle\frac{(v-U^n)^2}{2}f^n\rangle + \frac{Z}{2}N^nT_e^n \equiv \frac{N}{2}(T^0 + ZT_e^0)$ *do not depend on* n.

That is, the semi-implicit scheme is also totally conservative and consistent with (1.3), and the discrete temperature equation for the electrons has the same form as the explicit scheme (3.3); that is,

$$T_e^{n+1} = T_e^n\left(1 - \Delta t\frac{2\Omega^n}{Z}\right) + 2\Delta t\Omega^n\frac{\widetilde{NT}^{n+1/2}}{N^n}.$$

Conservation of the equilibrium state by the semi-implicit scheme. For this scheme, we cannot prove an H-theorem since we cannot obtain a formulation of $S(f^n, f^{n+1})_j$ equivalent to (A.9). However, we have the following result.

PROPOSITION 3.6. *The scheme defined by* (3.7) *and* (3.8) *preserves the equilibrium state; that is,*

$$f^n = f^\infty \quad \text{and} \quad T_e^n = T_e^\infty > 0 \quad \Longleftrightarrow \quad f^{n+1} = f^n \quad \text{and} \quad \forall j: f_j^n > 0,\ T_e^n > 0.$$

We recall that f^∞ is defined by the relation (2.18).

Proof of Proposition 3.6. Let us assume that

$$f^n = C\cdot\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \quad \text{and} \quad T_e^n = T_e^\infty.$$

Then, as for the proof of the relation (2.15), we get

$$\begin{aligned} &\forall\Delta t\ (f^{n+1} - C\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty})_j \\ &= \Delta t\frac{\Omega^nT_e^n}{m\Delta v^2}\Big[a_j(f^{n+1} - C\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty})_{j+1} - b_j(f^{n+1} - C\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty})_j \\ &\qquad + c_j(f^{n+1} - C\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty})_{j-1}\Big] \end{aligned}$$

with $C = \frac{N}{\langle \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \rangle}$. Since the diffusive matrix $\mathcal{D}$ (see (2.11)) is diagonally dominant, $\mathcal{I} + \Delta t \cdot \mathcal{D}$ is definite positive, which necessarily shows that

$$f^{n+1} = C \cdot \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} = f^n.$$

Now, by assuming that $f^{n+1} = f^n$, we can apply the equality (A.11) and we get

$$\sum_j S(f^n, f^n)_j \log(f/\mathbf{M}_{\widetilde{U}, T_e})_j^n$$

$$= -\frac{\Omega^n T_e^n}{m\Delta v^2} \sum_j \widetilde{f}_{j+1/2}^n \left[\log(f/\mathbf{M}_{\widetilde{U}, T_e})_{j+1}^n - \log(f/\mathbf{M}_{\widetilde{U}, T_e})_j^n \right]^2 = 0,$$

which shows that there exists C such that

$$f^n = C \cdot \mathbf{M}_{\widetilde{U}^n, T_e^n}$$

since, by hypothesis, $\widetilde{f}_{j+1/2}^n > 0$ and $\Omega^n T_e^n > 0$. To conclude, we have to say only that the solution of the system (2.20) is unique (see point (iii) of Proposition 2.7) and that $\mathbf{M}_{\widetilde{U}^n, T_e^n}$ is a solution of the system (2.20) since the scheme is conservative. $\square$

On the positivity of the semi-implicit scheme. We can remark that the semi-implicit scheme realizes a splitting between the convective and the diffusion parts of the operator. Then, by neglecting the corrective term $\delta\widetilde{u}^{n+1/2}$, we can formulate it as

$$\frac{1}{\Delta t}(f_j^{n+1/2} - f_j^n) = \frac{\Omega^n}{\Delta v} \left[(v_{j+1/2} - \widetilde{U}^n)\widetilde{f}_{j+1/2}^n - (v_{j-1/2} - \widetilde{U}^n)\widetilde{f}_{j-1/2}^n \right],$$

discretization of

$$\partial_t f = \Omega \partial_v \left[(v - U) f \right], \tag{3.10}$$

and

$$\frac{1}{\Delta t}(f_j^{n+1} - f_j^{n+1/2}) = \frac{\Omega^n T_e^n}{m\Delta v^2}(a_j f_{j+1}^{n+1} - b_j f_j^{n+1} + c_j f_{j-1}^{n+1}),$$

discretization of

$$\partial_t f = \frac{\Omega T_e}{m} \partial_{v^2} f. \tag{3.11}$$

It is clear that $f_j^{n+1/2} > 0 \Longrightarrow f_j^{n+1} > 0$ for any time step Δt since the inverse of a diagonal dominant matrix is a positive matrix. The difficulty arises for the convective part since we have to find a criteria $\Delta t \leq \Delta t^n$ such that $f_j^n > 0 \Longrightarrow f_j^{n+1/2} > 0$. Such a criteria with Δt^n not going toward 0 can be found only if $\inf_{j,n}(f_j^n)$ is strictly positive. In our case, we can expect only that we would have $\inf_{j,n}(f_j^n) > 0$ thanks to the smoothing effect of the diffusion operator, knowing that the solution of (3.10) is a classical Dirac measure when t goes to infinity; the numerical results seem to confirm this fact.

4. Numerical results. We test the numerical semi-implicit scheme. In the following test cases, the collision frequency Ω is given by (see [2], [3], or [4])

$$\Omega = \frac{4}{3}\sqrt{2\pi}\frac{\sqrt{m_e}NZ^3e^4\log\Lambda}{mT_e^{3/2}}, \tag{4.1}$$

where $\log\Lambda$ is the Coulombian logarithm which is supposed to be a constant and equal to 10. We recall that the numerical entropy is defined by

$$H(t) = \langle f\log f\rangle - \frac{ZN}{2}\log T_e.$$

We introduce a CFL number equal to

$$CFL = \Delta t/\frac{m\Delta v^2}{4\Omega^0 T_e^0}.$$

It is related to the discretization of the diffusion operator and then to the theoretical time step Δt_1^n given by (3.4). Moreover, we take

$$v_{j_{\max}} = -v_{1/2} = 5\sqrt{\frac{T_e^0}{m}} \quad \text{with} \quad T_e^0 = 2 \text{ KeV}$$

and

$$\Delta v = \frac{1}{10}\cdot\sqrt{\frac{T_e^0}{m}};$$

that is, $j_{\max} = 100$ (except for the test case 3 also tested with $j_{\max} = 10$). Then, we can rewrite the variable CFL as (when $j_{\max} = 100$)

$$\Delta t = \frac{CFL}{400}\cdot\frac{1}{\Omega^0}.$$

For the three first test cases, the results of our explicit and semi-implicit schemes (i.e., by using the entropic average) are quasi-identical because the CFL number is close to 1. In the fourth test case, we show that the semi-implicit scheme allows us to use very large time steps.

Test case 1. We consider the following initial conditions:

$$\begin{cases} f^0 = \text{bi-Maxwellian centered and of temperature } T^0, \\ T^0 = 1 \text{ KeV}, \\ T_e^0 = 2 \text{ KeV}, \\ U^0 = 0 \end{cases}$$

with

$$\begin{cases} \rho/m = 10^{22} \text{ cm}^{-3}, \\ m = 2.5m_p \text{ (where } m_p \text{ is the proton mass)}, \\ Z = 2, \end{cases}$$

and we take $CFL = 1.7$. We can remark on Figures 1, 2, and 3 that the relaxation has a good behavior. We can also remark that when $j_{\max} = 100$, the entropic average and the arithmetic average give very close results.

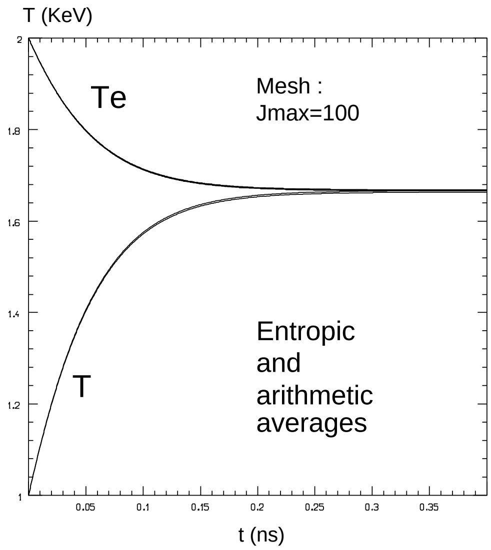

FIG. 1.

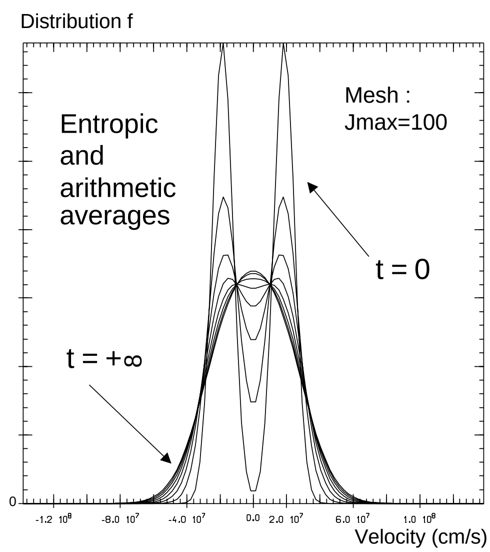

FIG. 2.

Test case 2. To study the conservation of the equilibrium state, we now consider a plasma whose initial conditions are

$$\begin{cases} f^0 = \text{centered Maxwellian with temperature } T^0, \\ T^0 = 1 \text{ KeV}, \\ T_e^0 = 1 \text{ KeV}, \\ U^0 = 0, \end{cases}$$

and we study the behavior of the scheme when $\widetilde{f}$ is defined by the entropic average, the classical Chang–Cooper average (see [8]), the arithmetic average, and the harmonic

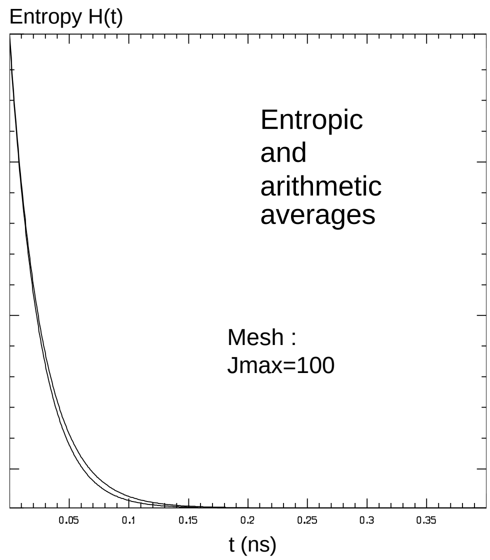

FIG. 3.

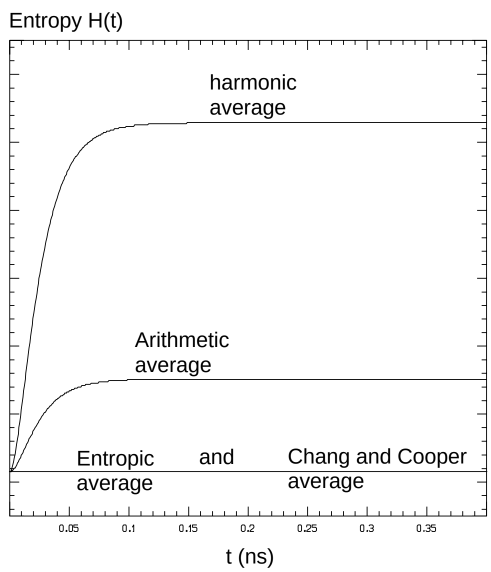

FIG. 4.

average (used in [17]) with $j_{\max} = 100$. Figure 4 shows that only the two first choices of $\widetilde{f}$ preserve the equilibrium state.

Test case 3. To study more finely the difference between all choices of $\widetilde{f}$, we consider the following initial conditions:

$$\left\{ \begin{array}{l} f^0 = \text{centered Maxwellian with temperature } T^0, \\ T^0 = 1 \text{ KeV}, \\ T_e^0 = 2 \text{ KeV}, \\ U^0 = 0 \end{array} \right.$$

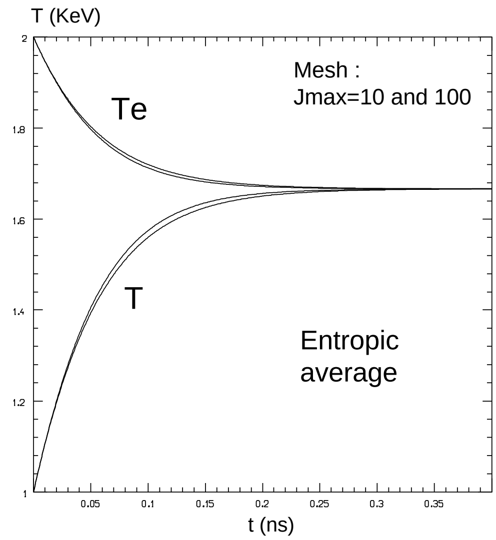

FIG. 5.

with

$$\begin{cases} \rho/m = 10^{22}\ \mathrm{cm}^{-3}, \\ m = 2.5 m_p \text{ (where } m_p \text{ is the mass of a proton)}, \\ Z = 2, \end{cases}$$

and we take $CFL = 1.7$. We also remark on Figures 5–13 that these four averages give similar results on a fine grid, but only the entropic and the Chang–Cooper ones give good relaxation that trends to the right equilibrium state and positivity of the solution. For $j_{\max} = 10$, we also verify that the scheme is entropic only for the entropic (see Figure 11) and Chang–Cooper (see Figure 12) averages. Figures 5, 7, 11, and 12 show also that the entropic averages give results that are better than the ones given by the Chang–Cooper average on a small grid.

Test case 4: *Study of the time step.* We can see in Figure 14 (see test case 3 for the initial conditions) that the semi-implicit scheme gives a good relaxation of the temperature even when the CFL factor is large (we have taken $CFL = 108$). However, we can also notice that for this test case, the value of the CFL factor to ensure the decay of the entropy is about of an order of 20 (with $j_{\max} = 100$). For the explicit scheme running on the same test case, the maximum CFL factor is about 1.7, independently of the velocity step Δv.

All the results of these test cases show us that the explicit and semi-implicit schemes using the entropic average allow us to simulate the ion/electron collision operator well even on a grid with very large Δv. Moreover, the semi-implicit scheme accepts very large time steps.

5. Conclusion. To solve the ion/electron Fokker–Planck equation in Cartesian geometry, we have proposed a numerical scheme of finite difference type based on the *entropic average*, an average which is introduced in this paper (see Definition 2.1). The semi-implicit scheme has been used in [19, Part 2, Chapter 4] for solving the full kinetic system (0.5) and (0.6). This scheme may obviously be extended to the dimension 2 or 3, and we have shown that it is a good alternative to the Chang–Cooper

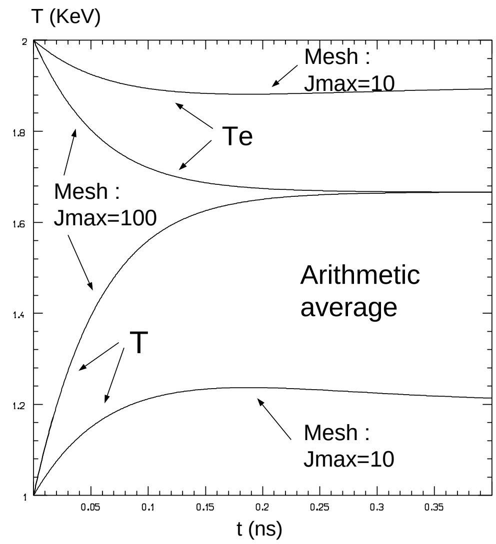

Fig. 6.

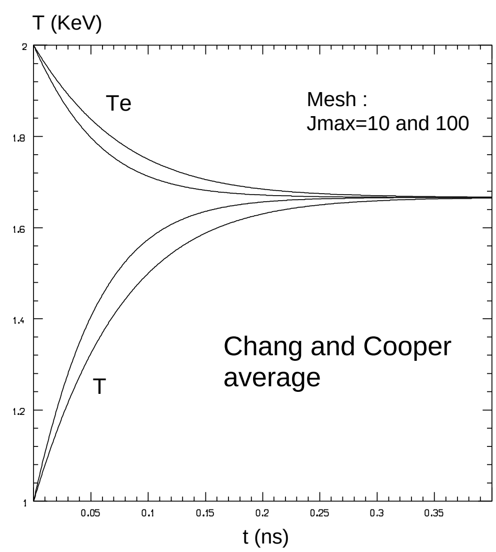

Fig. 7.

average: indeed, it always gives positive and precise solutions even on very coarse grids (see also [19, Part 2, Chapter 4]) for hard numerical applications coming from the inertial confinement fusion field) and it is simpler. Moreover, it is possible to recover the results of this paper in axisymmetrical or spherical geometries (see [19, Part 1, Chapter 4]), and the entropic average may also be used for the numerical treatment of other kinetic operators of Fokker–Planck type (see [15] and [20])—for instance, for the classical quadratic Fokker–Planck operator $B(f)$ in spherical geometry which is

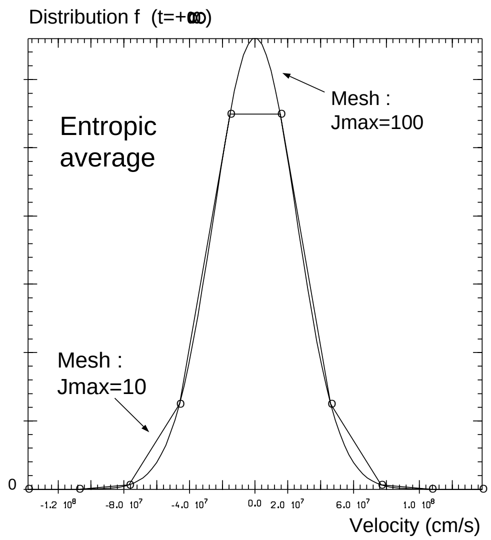

Fig. 8.

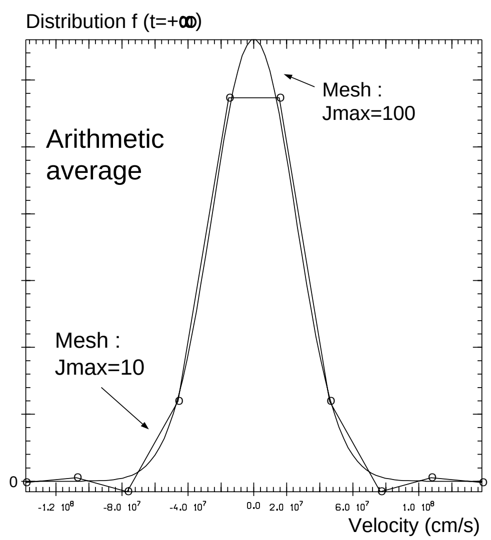

Fig. 9.

defined by (see [11])

$$B(f)(w) = w^{-1/2}\frac{\partial}{\partial w}\int_{\mathbf{R}^+}\left(f(w')\frac{\partial f}{\partial w}(w) - f(w)\frac{\partial f}{\partial w'}(w')\right)\min(w^{3/2}, w'^{3/2})dw'.$$

Here, w is the square of the modulus of the velocity. In this framework, the *entropic average* defines a scheme which is closely related to one of the schemes proposed in [11] (see [15]).

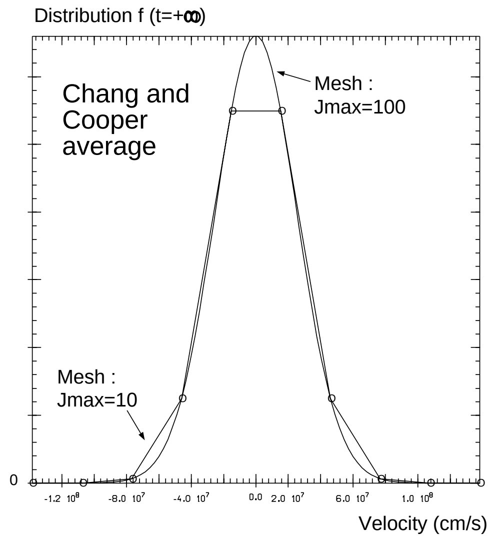

Fig. 10.

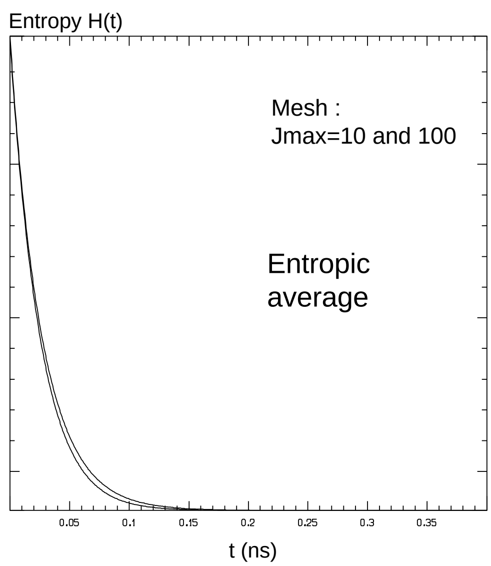

Fig. 11.

Appendix. Proof of Propositions 2.4, 2.7, 3.2, and 3.3.

A.1. Proof of Proposition 2.4. Let us remark that by applying the Cauchy–Peano theorem (or the Cauchy–Lipschitz theorem), we can say that there exists a maximum interval $[0, T[$ on which the semidiscretized scheme (2.7) and (2.8) admits a strictly positive solution. (We recall that the initial conditions are strictly positive.) Let us note that $H(f, T_e)(t)$ is defined on $[0, T[$ and that, due to Lemma 2.3, we have $H(f, T_e)(t) \leq H(f, T_e)(0) < +\infty$ on $[0, T[$; then, since $x \mapsto x \log x$ is bounded from below on $[0, +\infty[$, we have $\inf_{t \in [0,T[} T_e(t) > 0$ (otherwise, $\lim_{t \to T} H(f, T_e)(t) = +\infty$) and, consequently, $\inf_{t \in [0,T[} \Omega(t) > 0$.

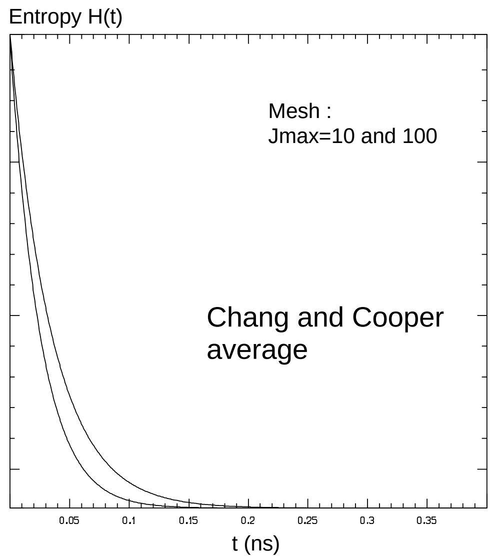

FIG. 12.

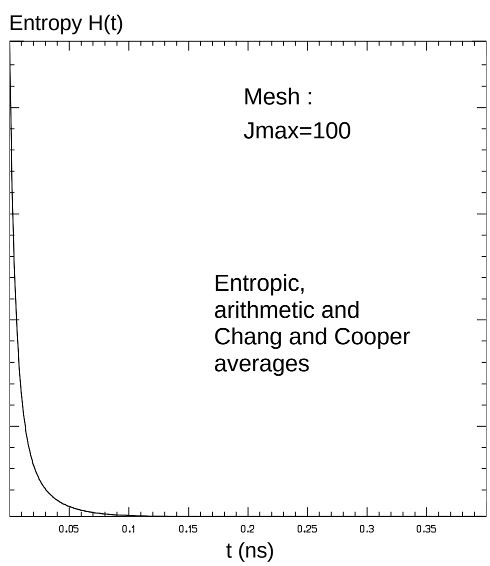

FIG. 13.

If $T < +\infty$, necessarily, $\exists j_0 / \lim_{t\to T} f_{j_0}(t) = 0$ which implies that $\lim_{t\to T} \widetilde{f}_{j_0\pm 1/2}(t) = 0$. When $\lim_{t\to T} \widetilde{N}(t) > 0$, we have $\sup_{t\in[0,T[} |\widetilde{U}(t)| < +\infty$; and when $\lim_{t\to T} \widetilde{N}(t) = 0$ (i.e., $\forall j : \lim_{t\to T} \widetilde{f}_{j+1/2}(t) = 0$ which does not mean that $\forall j : \lim_{t\to T} f_j(t) = 0$), by continuity, we obtain $\lim_{t\to T} \widetilde{U}(t) = 0$. Then, in each case, when $T < +\infty$, the convective part of (2.8) goes to zero when t goes to T, which shows that

$$\lim_{t\to T} \partial_t f_{j_0}(t) = \lim_{t\to T} \frac{\Omega(t) T_e(t)}{m\Delta v^2} \left[f_{j_0+1}(t) + f_{j_0-1}(t) \right] \geq 0.$$

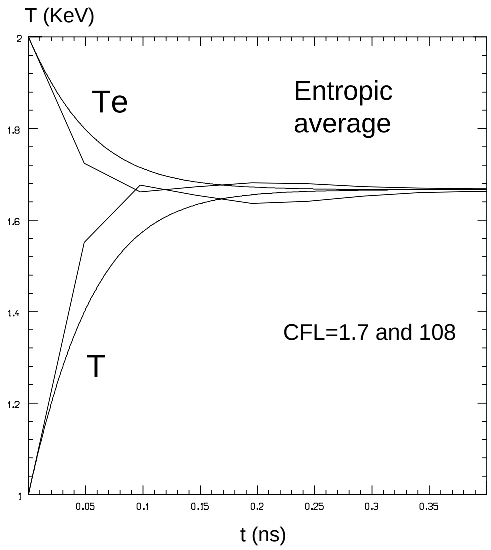

FIG. 14.

On the other hand, since $f_{j_0}(t)$ is strictly positive for $t < T$, necessarily, we must have

$$\lim_{t\to T} \partial_t f_{j_0}(t) \le 0.$$

Thus, $\lim_{t\to T} \partial_t f_{j_0}(t) = 0$, and then $\lim_{t\to T} f_{j_0+1}(t) = \lim_{t\to T} f_{j_0-1}(t) = 0$ since $\inf_{t\in[0,T[} T_e(t) > 0$ and $\inf_{t\in[0,T[} \Omega(t) > 0$. By continuation, we obtain that

$$T < +\infty \Longrightarrow \forall j : \lim_{t\to T} f_j(t) = 0,$$

which is impossible since we know that for all $t \in [0, T[: \sum_j f_j(t)\Delta v = N(t) = N(0) > 0$ (see Proposition 2.2).

Then, the only possibility is that the maximal interval is equal to $[0, +\infty[$. And, finally, we can write that $\inf_{t\in[0,+\infty[} T_e(t) > 0$.

A.2. Proof of Proposition 2.7. Let us recall the notation

$$\mathbf{M}_{U,T}(v,t) = \frac{N}{\sqrt{2\pi T/m}} \exp\left[-\frac{m(v-U)^2}{2T}\right].$$

A.2.1. Preliminary results: Lemmas A.1, A.2, and A.3. The proof of Proposition 2.7 uses the following lemma.

LEMMA A.1. *The relation*

$$H(f, T_e) - H(f^\infty, T_e^\infty) = \sum_j f_j \log\left(\frac{f_j}{f_j^\infty}\right)\Delta v + Z\int \mathbf{M}_{\widetilde{U}^\infty,T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty,T_e}}{\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}}\right) dv$$

is verified.

LEMMA A.2. *For all sequence (t_k) such that*

$$\text{(A.1)}\quad \lim_{t_k\to+\infty} \max_j \left(\widetilde{f}_{j+1/2}(t_k)\left[\log(f/\mathbf{M}_{\widetilde{U},T_e})_{j+1}(t_k) - \log(f/\mathbf{M}_{\widetilde{U},T_e})_j(t_k)\right]^2\right) = 0,$$

it exists a subsequence, still denoted (t_k), *such that*

(i)

$$\inf_k \left|\widetilde{N}(t_k)\right| > 0, \quad \sup_k \left|\widetilde{U}(t_k)\right| < +\infty \quad \textit{and} \quad \sup_k |\delta\widetilde{u}(t_k)| < +\infty,$$

(ii)

$$\inf_{k,j} f_j(t_k) > 0. \tag{A.2}$$

The proof of Lemma A.2 is based on the following lemma.

LEMMA A.3. *Let us define two sequences* x_k *and* y_k *of positives reals,* $\widetilde{f}_k$ *their entropic average, and* z_k *a real sequence. And let us suppose that*

$$\forall k : \widetilde{f}_k[\log x_k - \log y_k - z_k]^2 \leq C.$$

Then, the two following properties are verified:

(i) *If for one* C' *positive we have for all* k

$$z_k > -C',$$

then

$$x_k \to 0 \Longrightarrow y_k \to 0.$$

(ii) *If* z_k *is bounded, then*

$$\widetilde{f}_k \to 0 \Longrightarrow x_k \to 0 \textit{ and } y_k \to 0.$$

Proof of Lemma A.1. We can write

$$\begin{aligned}
&H(f,T_e) - H(f^\infty, T_e^\infty) \\
&\quad = \sum_j f_j \log f_j \Delta v + Z \int \mathbf{M}_{\widetilde{U}^\infty, T_e} \log \mathbf{M}_{\widetilde{U}^\infty, T_e} dv \\
&\qquad - \sum_j f_j^\infty \log f_j^\infty \Delta v - Z \int \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \log \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} dv \\
&\quad = \sum_j f_j \log\left(\frac{f_j}{f_j^\infty}\right) \Delta v + Z \int \mathbf{M}_{\widetilde{U}^\infty, T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty, T_e}}{\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}}\right) dv \\
&\qquad + \left[\sum_j \left(f_j - f_j^\infty\right) \log f_j^\infty \Delta v + Z \int \left(\mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}\right) \log \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} dv\right].
\end{aligned}$$

Since this scheme conserves the mass, we have

$$\sum_j \left(f_j - f_j^\infty\right) \log f_j^\infty \Delta v = \frac{1}{T_e^\infty} \sum_j \frac{m}{2}\left(v_j - \widetilde{U}^\infty\right)^2 \left(f_j^\infty - f_j\right) \Delta v.$$

Then

$$\sum_j \left(f_j - f_j^\infty\right) \log f_j^\infty \Delta v = \frac{1}{T_e^\infty} \sum_j \frac{m}{2} \left(v_j - U^0\right)^2 \left(f_j^\infty - f_j\right) \Delta v$$

$$+ \frac{m\left(U^0 - \widetilde{U}^\infty\right)}{2T_e^\infty} \left[\left(U^0 - \widetilde{U}^\infty\right) \sum_j \left(f_j^\infty - f_j\right) \Delta v + 2 \sum_j \left(v_j - U^0\right) \left(f_j^\infty - f_j\right) \Delta v\right].$$

However, we have $\sum_j \left(f_j^\infty - f_j\right) \Delta v = 0$ and $\sum_j \left(v_j - U^0\right) \left(f_j^\infty - f_j\right) \Delta v = 0$ since the scheme conserves the mass and the momentum, and since $\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}$ is a solution of the system (2.20). Then we can write

$$\sum_j \left(f_j - f_j^\infty\right) \log f_j^\infty \Delta v = \frac{1}{T_e^\infty} \sum_j \frac{m}{2} \left(v_j - U^0\right)^2 \left(f_j^\infty - f_j\right) \Delta v.$$

And since

$$\int \left(\mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}\right) \log \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} dv = -\frac{N}{2T_e^\infty} \left(T_e - T_e^\infty\right),$$

in conclusion we have

$$H(f, T_e) - H(f^\infty, T_e^\infty) = \sum_j f_j \log\left(\frac{f_j}{f_j^\infty}\right) \Delta v + Z \int \mathbf{M}_{\widetilde{U}^\infty, T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty, T_e}}{\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}}\right) dv$$

$$+ \frac{1}{T_e^\infty} \left[\sum_j \frac{m}{2} \left(v_j - U^0\right)^2 \left(f_j^\infty - f_j\right) \Delta v + \frac{ZN}{2} \left(T_e^\infty - T_e\right)\right].$$

However, the last term of the right-hand side is equal to zero since the scheme conserves the energy and since $\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}$ is a solution of the system (2.20). Then, we obtain the result. $\square$

Proof of Lemma A.2. (i) If there exists a subsequence of (t_k) such that

$$\lim_{t_k \to +\infty} \widetilde{N}(t_k) = 0, \tag{A.3}$$

then, for any j, we get

$$\lim_{t_k \to +\infty} \widetilde{f}_{j+1/2}(t_k) = 0.$$

Since $N(t_k) = N(0)$, there is at least one j such that (up to an extraction)

$$\inf_k f_j(t_k) > 0. \tag{A.4}$$

Moreover, up to an extraction, $\widetilde{U}(t_k)$ is bounded from below or bounded from above; let us suppose that $\widetilde{U}(t_k)$ is bounded from below. (Proof is similar if we suppose that $\widetilde{U}(t_k)$ is bounded from above.) Since $\widetilde{f}_{j+1/2}(t_k)$ converges to 0 and since $\inf_k f_j(t_k) > 0$, we deduce that $f_{j+1}(t_k)$ converges to 0. The relation (A.1) can be also written as

$$\lim_{t_k \to +\infty} \widetilde{f}_{j+1/2}(t_k)[\log f_{j+1}(t_k) - \log f_j(t_k) - z_k]^2 = 0 \qquad \text{with}$$

$$z_k = -\frac{m\Delta v}{T_e(t)} \left(v_{j+1/2} - \widetilde{U}(t_k)\right). \tag{A.5}$$

And since $\inf_k T_e(t_k) > 0$ (see Proposition 2.4), we can see that there exists $C' > 0$ such that $z_k > -C'$; the hypothesis of Lemma A.3(i) is verified with $x_k = f_{j+1}(t_k)$ and $y_k = f_j(t_k)$. Then, $\lim_{t_k \to +\infty} f_j(t_k) = 0$, which is in contradiction with (A.4). Thus, (A.3) is false and we have

$$\inf_k \left| \widetilde{N}(t_k) \right| > 0.$$

Knowing that

$$v_{1/2} + \frac{T_e(t)}{m} \cdot \frac{(f_{j_{\max}}(t) - f_1(t))}{\widetilde{N}(t)} \leq \widetilde{U}(t) \leq v_{j_{\max}+1/2} + \frac{T_e(t)}{m} \cdot \frac{(f_{j_{\max}}(t) - f_1(t))}{\widetilde{N}(t)},$$

we also obtain

$$\sup_k \left| \widetilde{U}(t_k) \right| < +\infty.$$

(Δv is fixed and $N(t) < +\infty$; then, for all $j, t \geq 0 : f_j(t) < +\infty$.) Using the definition of $\delta\widetilde{u}(t)$, we also have

$$\sup_k |\delta\widetilde{u}(t_k)| < +\infty.$$

(ii) If (A.2) is false, there exists a j_0 such that, up to an extraction, $\lim_{t_k \to +\infty} f_{j_0}(t_k) = 0$. Then, we have $\lim_{t_k \to +\infty} \widetilde{f}_{j_0 \pm 1/2}(t_k) = 0$. Since the relation (A.5) is true with z_k bounded, point (ii) of Lemma A.3 claims that also

$$\lim_{t_k \to +\infty} f_{j_0-1}(t_k) = 0 \quad \text{and} \quad \lim_{t_k \to +\infty} f_{j_0+1}(t_k) = 0,$$

which gives

$$\lim_{t_k \to +\infty} \widetilde{f}_{j_0-3/2}(t_k) = 0 \quad \text{and} \quad \lim_{t_k \to +\infty} \widetilde{f}_{j_0+3/2}(t_k) = 0.$$

By continuation, we deduce that for any j, $\lim_{t_k \to +\infty} f_j(t_k) = 0$ and $\lim_{t_k \to +\infty} N(t_k) = 0$ which is in contradiction with the conservation of the mass. $\square$

Proof of Lemma A.3. (i) We suppose that there exists a subsequence of y_k, also called y_k, which is bounded below by a constant $\alpha > 0$. Then, there is a contradiction since

$$\begin{aligned} \widetilde{f}_k[\log x_k - \log y_k - z_k]^2 &= (x_k - y_k)[\log x_k - \log y_k] + \widetilde{f}_k {z_k}^2 - 2z_k(x_k - y_k) \\ &\geq -y_k[\log x_k - \log y_k] + 2z_k(y_k - x_k) + o(1) \to +\infty. \end{aligned}$$

(ii) This point is a consequence of the first point since x_k and y_k play symmetrical roles. $\square$

A.2.2. Proof of point (i) of Proposition 2.7. Since $H(f, T_e)$ is continuous, decreasing, and bounded from below (see Lemma 2.3 and Corollary 2.5), and since $\inf_{t \in [0,+\infty[} T_e(t) > 0$ (see Proposition 2.4), the relation (2.16) shows that there exists a sequence (t_k) going to $+\infty$ and such that

$$\text{(A.6)} \quad \lim_{t_k \to +\infty} \max_j \left(\widetilde{f}_{j+1/2}(t_k) \left[\log(f/\mathbf{M}_{\widetilde{U},T_e})_{j+1}(t_k) - \log(f/\mathbf{M}_{\widetilde{U},T_e})_j(t_k) \right]^2 \right) = 0.$$

By using point (ii) of Lemma A.2, we know that up to an extraction of a subsequence

$$\inf_{k,j} \widetilde{f}_{j+1/2}(t_k) > 0$$

(except for $j = 0$ and $j = j_{\max}$ because of the boundary limits (2.12)). And using (A.6), we obtain

$$\lim_{t_k \to +\infty} \max_j \left[\log(f/\mathbf{M}_{\widetilde{U},T_e})_{j+1}(t_k) - \log(f/\mathbf{M}_{\widetilde{U},T_e})_j(t_k) \right] = 0,$$

which implies that there exists $C > 0$ such that

$$\lim_{t_k \to +\infty} \frac{f_j(t_k)}{\mathbf{M}_{\widetilde{U},T_e,j}(t_k)} = C.$$

By applying point (i) of Lemma A.2, we see that $\widetilde{U}(t_k)$ is bounded. Moreover, $T_e(t)$ is also bounded since the scheme conserves the energy. Then, we can state that there exists $T_e^\infty > 0$ (since $T_e(t)$ is also bounded from below; see Proposition 2.4), $\widetilde{U}^\infty$, and a subsequence of (t_k) still noted (t_k) such that

$$\lim_{t_k \to +\infty} T_e(t_k) = T_e^\infty > 0 \quad \text{and} \quad \lim_{t_k \to +\infty} \widetilde{U}(t_k) = \widetilde{U}^\infty,$$

which gives

$$\lim_{t_k \to +\infty} \mathbf{M}_{\widetilde{U},T_e,j}(t_k) = \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty,j}$$

and then

$$\lim_{t_k \to +\infty} f_j(t_k) = C \cdot \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty,j}.$$

Since the scheme conserves the mass and the energy, $(\widetilde{U}^\infty, T_e^\infty)$ is a solution of (2.20) and we have

$$C = N/\langle \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty} \rangle.$$

A.2.3. Proof of point (ii) of Proposition 2.7. Let $(\widetilde{U}^\infty, T_e^\infty)$ be a solution of the system (2.20) and let us recall Jensen's inequality (see [21] and [22]) which says that

$$\int g[w(v)] d\mu(v) \geq \int \mu(v) dv \cdot g\left[\frac{\int w(v) d\mu(v)}{\int \mu(v) dv} \right]$$

for each convex function $g(w)$ and finite positive measure $d\mu(v)$. Then, by applying this inequality with the function $g(w) = w \log w$ and the measure $\sum_j f_j^\infty \delta(v_j)$ (δ being the classical Dirac measure) or the measure $\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty} dv$, we get

$$\sum_j f_j \log\left(\frac{f_j}{f_j^\infty} \right) \Delta v > 0 \quad \text{and} \quad \int \mathbf{M}_{\widetilde{U}^\infty,T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty,T_e}}{\mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}} \right) dv > 0$$

since

$$\frac{\sum_j f_j \Delta v}{\sum_j f_j^\infty \Delta v} = 1 \qquad \text{and} \qquad \frac{\int \mathbf{M}_{\widetilde{U}^\infty,T_e}(v) dv}{\int \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty}(v) dv} = 1.$$

To conclude, we have to use only Lemma A.1.

A.2.4. Proof of point (iii) of Proposition 2.7. To obtain the unicity, we now apply the Csiszar–Kullback inequality (cf. [21] and [22]), which gives a better result for the convergence toward the equilibrium state

$$\left\| f_j - f_j^\infty \right\|_{l_1}^2 \leq 2 \sum_j f_j \log\left(\frac{f_j}{f_j^\infty}\right) \Delta v$$

and

$$\left\| \mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \right\|_{L_1}^2 \leq 2 \int \mathbf{M}_{\widetilde{U}^\infty, T_e} \log\left(\frac{\mathbf{M}_{\widetilde{U}^\infty, T_e}}{\mathbf{M}_{\widetilde{U}^\infty, T_e^\infty}}\right) dv.$$

Then

$$\left\| f_j - f_j^\infty \right\|_{l_1}^2 + Z \left\| \mathbf{M}_{\widetilde{U}^\infty, T_e} - \mathbf{M}_{\widetilde{U}^\infty, T_e^\infty} \right\|_{L_1}^2 \leq 2[H(f, T_e) - H(f^\infty, T_e^\infty)]$$

by using Lemma A.1. Thus, if the system (2.20) has two solutions $\mathbf{M}_{\widetilde{U}^{\infty,1}, T_e^{\infty,1}}$ and $\mathbf{M}_{\widetilde{U}^{\infty,2}, T_e^{\infty,2}}$ with

$$\mathbf{M}_{\widetilde{U}^{\infty,k}, T_e^{\infty,l}}(v,t) = \frac{N}{\sqrt{2\pi T_e^{\infty,l}/m}} \exp\left[-\frac{m(v - \widetilde{U}^{\infty,k})^2}{2T_e^{\infty,l}}\right]$$

($k, l \in \{1, 2\}$), we have

$$H(f^{\infty,2}, T_e^{\infty,2}) - H(f^{\infty,1}, T_e^{\infty,1}) \geq 0 \quad \text{and} \quad H(f^{\infty,1}, T_e^{\infty,1}) - H(f^{\infty,2}, T_e^{\infty,2}) \geq 0,$$

which gives

$$H(f^{\infty,2}, T_e^{\infty,2}) - H(f^{\infty,1}, T_e^{\infty,1}) = 0.$$

Then, we have

$$\text{(A.7)} \qquad \left\| f_j^{\infty,2} - f_j^{\infty,1} \right\|_{l_1}^2 = 0 \quad \text{and} \quad \left\| \mathbf{M}_{\widetilde{U}^{\infty,k}, T_e^{\infty,l}} - \mathbf{M}_{\widetilde{U}^{\infty,k}, T_e^{\infty,k}} \right\|_{L_1}^2 = 0.$$

And the first part of (A.7) gives

$$\text{(A.8)} \qquad \forall j : f_j^{\infty,2} = f_j^{\infty,1}.$$

The equality (A.8) shows that $\widetilde{U}^{\infty,2} = \widetilde{U}^{\infty,1}$ which implies that $\mathbf{M}_{\widetilde{U}^{\infty,k}, T_e^{\infty,l}} = \mathbf{M}_{\widetilde{U}^{\infty,l}, T_e^{\infty,l}}$. Then, using the second part of (A.7), we obtain $\mathbf{M}_{\widetilde{U}^{\infty,1}, T_e^{\infty,1}} = \mathbf{M}_{\widetilde{U}^{\infty,2}, T_e^{\infty,2}}$ and then $T_e^{\infty,2} = T_e^{\infty,1}$ which gives us the unicity of the solution of the system (2.20).

A.3. Proof of Proposition 3.2. • As for the semidiscretized problem, we show that

$$\text{(A.9)} \qquad S(f^n)_j = \frac{\Omega^n T_e^n}{m \Delta v^2} \left(k_{j+1/2} + k_{j-1/2}\right)$$

with

$$k_{j+1/2} = \widetilde{f}_{j+1/2}^n \left[\log(f_{j+1}^n / f_j^n) - \log(\mathbf{M}_{\widetilde{U}^n, T_e^n, j+1} / \mathbf{M}_{\widetilde{U}^n, T_e^n, j})\right].$$

If $k_{j+1/2}$ and $k_{j-1/2}$ are positive, f_j^{n+1} will be positive. We restrict the study for the more restrictive situation which corresponds to

$$k_{j+1/2} < 0 \qquad \text{and} \qquad k_{j-1/2} < 0,$$

which is equivalent to

$$f_{j+1}^n/f_j^n < \mathbf{M}_{\widetilde{U}^n,T_e^n,j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j} \qquad \text{and} \qquad f_{j-1}^n/f_j^n < \mathbf{M}_{\widetilde{U}^n,T_e^n,j-1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j}.$$

We now suppose that $f_{j+1}^n \neq f_j^n$. Then, we can write

$$k_{j+1/2} = f_j^n \left[\left(f_{j+1}^n/f_j^n - 1\right) - \frac{\left(f_{j+1}^n/f_j^n - 1\right)}{\log(f_{j+1}^n/f_j^n)} \cdot \log(\mathbf{M}_{\widetilde{U}^n,T_e^n,j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j}) \right].$$

Since $x \mapsto \frac{x-1}{\log x}$ is continuous, positive, and increasing on $\mathbf{R}^+$ and since $f_{j+1}^n/f_j^n < \mathbf{M}_{\widetilde{U}^n,T_e^n j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j}$, we get

$$\begin{aligned}
\left|k_{j+1/2}\right| &\leq f_j^n \left[\left|f_{j+1}^n/f_j^n - 1\right| + \left|\mathbf{M}_{\widetilde{U}^n,T_e^n,j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j} - 1\right| \right] \\
&\leq f_j^n \left[\max\left(1, f_{j+1}^n/f_j^n\right) + \max\left(1, \mathbf{M}_{\widetilde{U}^n,T_e^n,j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j}\right) \right] \\
&\leq 2 f_j^n \max\left(1, \mathbf{M}_{\widetilde{U}^n,T_e^n,j+1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,j}\right) \\
&\leq 2 f_j^n \max_i \left(\mathbf{M}_{\widetilde{U}^n,T_e^n,i\pm1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,i}\right).
\end{aligned}$$

This is still true when $f_{j+1}^n = f_j^n$. Now, according to (A.9), we obtain that

$$f_j^{n+1} > f_j^n \left[1 - \frac{4\Omega^n T_e^n \Delta t}{m \Delta v^2} \max_i \left(\mathbf{M}_{\widetilde{U}^n,T_e^n,i\pm1}/\mathbf{M}_{\widetilde{U}^n,T_e^n,i}\right) \right]. \tag{A.10}$$

This is still true if $k_{j+1/2}$ or $k_{j-1/2}$ is positive. Therefore

$$\Delta t < \Delta t_1^n \Longrightarrow \forall j,\ f_j^{n+1} > 0.$$

- The positivity of T_e^{n+1} when $\Delta t < \Delta t_2^n$ is clear if we use the expression (3.3) and the definition of $\widetilde{NT}^n$.
- To prove the H-theorem, we use the relation (A.9) which allows us to write

$$\begin{aligned}
\sum_j S(f^n)_j \log(f/\mathbf{M}_{\widetilde{U},T_e})_j^n = -\frac{\Omega^n T_e^n}{m\Delta v^2} \sum_j \widetilde{f}_{j+1/2}^n [&\log(f/\mathbf{M}_{\widetilde{U},T_e})_{j+1}^n \\
&- \log(f/\mathbf{M}_{\widetilde{U},T_e})_j^n]^2 \leq 0.
\end{aligned} \tag{A.11}$$

- Let us now show the conservation of the equilibrium state. We suppose that

$$f^n = \frac{N}{\langle \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty} \rangle} \mathbf{M}_{\widetilde{U}^\infty,T_e^\infty} \quad \text{and} \quad T_e^n = T_e^\infty.$$

By construction, we have (see the system (2.20))

$$\langle f^n \rangle = N^0, \quad \langle (v - U^0) f^n \rangle = 0$$

and

$$\left\langle \left\{ \left[\frac{1}{2} m(v - U^0)^2 + Z \frac{T_e^n}{2} \right] - \left[\frac{T^0}{2} + Z \frac{T_e^0}{2} \right] \right\} \cdot f^n \right\rangle = 0$$

since $T_e^n = T_e^\infty$. Using the boundary conditions $\widetilde{f}_{1/2}^n = \widetilde{f}_{j_{\max}+1/2}^n \equiv 0$, we can also verify that $\widetilde{U}^n = \widetilde{U}^\infty$. Then,

$$\exists C > 0 / f^n = C \cdot \mathbf{M}_{\widetilde{U}^n, T_e^n}.$$

And, according to (A.9), we get for all $j : S(f^n)_j = 0$; that is to say $f^{n+1} = f^n$. For the converse, if for all $j : S(f^n)_j = 0$, then, according to (A.11), we get

$$\forall j : \log(f/\mathbf{M}_{\widetilde{U},T_e})_{j+1}^n = \log(f/\mathbf{M}_{\widetilde{U},T_e})_j^n$$

since $\widetilde{f}_{j+1/2}^n > 0$ (for $j \in \{2, \ldots, j_{\max} - 1\}$) and $\Omega^n T_e^n > 0$ by hypothesis. This shows that

$$\exists C > 0 / f^n = C \cdot \mathbf{M}_{\widetilde{U}^n, T_e^n}.$$

To conclude, we have to say only that the solution of the system (2.20) is unique (see point (iii) of Proposition 2.7) and that $\mathbf{M}_{\widetilde{U}^n, T_e^n}$ is a solution of the system (2.20) since the scheme is conservative.

A.4. Proof of Proposition 3.3.

A.4.1. Preliminary result: Lemma A.4. The proof of Proposition 3.3 uses the following lemma.

LEMMA A.4. *When $\widetilde{f}_{j+1/2}^n$ is the entropic average of f_j^n and of f_{j+1}^n, the inequality*

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq -\frac{4\Omega^n T_e^n}{m\Delta v^2} \cdot \mathfrak{M}^n \frac{h_{\max}^n}{h_{\min}} \sum_j S(f^n)_j \log \left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}} \right)_j$$

is verified.

Proof of Lemma A.4. By applying Schwarz's inequality, we obtain

$$S(f^n)_j^2 \leq \frac{\Omega^n T_e^n}{m\Delta v^2} \left(\widetilde{f}_{j+1/2}^n + \widetilde{f}_{j-1/2}^n \right) \cdot \frac{\Omega^n T_e^n}{m\Delta v^2} \Big\{ \widetilde{f}_{j+1/2}^n \Big[\log \left(h_{j+1}^n / h_j^n \right) \Big]^2 \\ + \widetilde{f}_{j-1/2}^n \Big[\log \left(h_{j-1}^n / h_j^n \right) \Big]^2 \Big\},$$

where $h_j^n = f_j^n / \mathbf{M}_{\widetilde{U}^n, T_e^n, j}$. Moreover, we can verify that

$$\frac{f_{j\pm1}^n}{f_j^n} \leq \frac{\mathbf{M}_{\widetilde{U}^n, T_e^n, j\pm1}}{\mathbf{M}_{\widetilde{U}^n, T_e^n, j}} \cdot \frac{h_{\max}^n}{h_{\min}^n} \leq \mathfrak{M}^n \cdot \frac{h_{\max}^n}{h_{\min}^n}.$$

Then

$$\max_j \left(\frac{\widetilde{f}_{j+1/2}^n + \widetilde{f}_{j-1/2}^n}{f_j^n} \right) \leq \frac{2\,(\mathfrak{M}^n h_{\max}^n / h_{\min}^n - 1)}{\log\,(\mathfrak{M}^n h_{\max}^n / h_{\min}^n)} \tag{A.12}$$

since $\frac{x-1}{\log x}$ is an increasing function when $x > 0$. Finally, we obtain

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq \frac{\Omega^n T_e^n}{m\Delta v^2} \cdot \frac{2\,(\mathfrak{M}^n h_{\max}^n/h_{\min}^n - 1)}{\log\,(\mathfrak{M}^n h_{\max}^n/h_{\min}^n)} \cdot \frac{2\Omega^n T_e^n}{m\Delta v^2} \sum_j \widetilde{f}_{j+1/2}^n [\log\left(h_{j+1}^n/h_j^n\right)]^2.$$

On the other hand, due to the equalities (2.14) and (2.16), we know that

$$\sum_j S(f^n)_j \log(f^n/\mathbf{M}_{\widetilde{U}^n,T_e^n})_j = -\frac{\Omega^n T_e^n}{m\Delta v^2} \sum_j \widetilde{f}_{j+1/2}^n [\log\left(h_{j+1}^n/h_j^n\right)]^2,$$

which give us

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq -\frac{4\Omega^n T_e^n}{m\Delta v^2} \cdot \frac{\mathfrak{M}^n h_{\max}^n/h_{\min}^n - 1}{\log\,(\mathfrak{M}^n h_{\max}^n/h_{\min}^n)} \sum_j S(f^n)_j \log\left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n,T_e^n}}\right)_j.$$

Then, if we remark that $\frac{x-1}{\log x} \leq \max\,(x,1)$ when $x > 0$, we get the result. $\square$

A.4.2. Proof of Proposition 3.3. Using the inequality for all $x : \log(1+x) \leq x$, we obtain

$$\sum_j \left[f^{n+1}\log(f^{n+1})\right]_j \leq \sum_j \left[f^n \log(f^n)\right]_j + \Delta t \sum_j \left[S(f^n)\log(f^n) + \Delta t \frac{S(f^n)^2}{f^n}\right]_j;$$

that is,

$$H^{n+1} \leq \langle f^n \log(f^n)\rangle + \Delta t \left\langle S(f^n)\log(f^n) + \Delta t \frac{S(f^n)^2}{f^n}\right\rangle - \frac{ZN^n}{2}\log(T_e^{n+1}). \tag{A.13}$$

On the other side, we have

$$T_e^{n+1} = T_e^n \left(1 - \frac{\Delta t}{ZN^n T_e^n} m\langle v_j^2 S(f^n)\rangle\right). \tag{A.14}$$

Using (2.13), we verify that

$$\langle S(f^n)\log(\mathbf{M}_{\widetilde{U}^n,T_e^n})\rangle = -\frac{m}{2T_e^n}\langle v^2 S(f^n)\rangle,$$

which allows us to claim that

$$T_e^{n+1} = T_e^n \left(1 + \frac{2\Delta t}{ZN^n}\left\langle S(f^n)\log\left(\mathbf{M}_{\widetilde{U}^n,T_e^n}\right)\right\rangle\right) \tag{A.15}$$

using (A.14). Then, by putting (A.15) in (A.13), we find

$$\begin{aligned} H^{n+1} \leq & H^n + \Delta t \left\langle S(f^n)\log(f^n) + \Delta t \frac{S(f^n)^2}{f^n}\right\rangle \\ & - \frac{ZN^n}{2}\log\left[1 + \frac{2\Delta t}{ZN^n}\sum_j S(f_j^n)\log(\mathbf{M}_{\widetilde{U}^n,T_e^n})_j \Delta v\right]. \end{aligned}$$

We remark that

$$\Delta t < \Delta t_2^n \Longrightarrow T_e^{n+1} > 0 \Longrightarrow \frac{2\Delta t}{ZN^n}\sum_j S(f_j^n)\log(\mathbf{M}_{\widetilde{U}^n,T_e^n})_j \Delta v > -1.$$

We deduce that

$$2\Delta t < \Delta t_2^n \Longrightarrow \frac{2(2\Delta t)}{ZN^n} \sum_j S(f_j^n) \log(\mathbf{M}_{\widetilde{U}^n, T_e^n})_j \Delta v > -1;$$

that is,

$$\Delta t < \Delta t_4^n = \Delta t_2^n/2 \Longrightarrow T_e^{n+1} > 0 \quad \text{and} \quad \frac{2\Delta t}{ZN^n} \sum_j S(f_j^n) \log(\mathbf{M}_{\widetilde{U}^n, T_e^n})_j \Delta v > -\frac{1}{2}.$$

On the other side, we easily verify that

$$\forall x > -\frac{1}{2} : \log\left(\frac{1}{1+x}\right) < x(2x-1).$$

And, by setting $x = \frac{2\Delta t}{ZN^n} \sum_j S(f_j^n) \log(\mathbf{M}_{\widetilde{U}^n, T_e^n})_j \Delta v$, we obtain that

$$\begin{aligned} H^{n+1} \leq & H^n + \Delta t \sum_j S(f_j^n) \log\left(\frac{f_j^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}}\right)_j \Delta v + \Delta t \frac{S(f_j^n)^2}{f_j^n} \Delta v \\ & + \frac{4\Delta t^2}{ZN^n} \left[\sum_j S(f_j^n) \log(\mathbf{M}_{\widetilde{U}^n, T_e^n})_j \Delta v\right]^2. \end{aligned}$$

Using Schwarz's inequality, we can also write

$$\begin{aligned} \frac{4\Delta t^2}{ZN^n} \left[\sum_j S(f_j^n) \log(\mathbf{M}_{\widetilde{U}^n, T_e^n})_j \Delta v\right]^2 &= \frac{4\Delta t^2}{ZN^n} \left[\sum_j S(f_j^n) \frac{m(v_j - \widetilde{U}^n)^2}{2T_e^n} \Delta v\right]^2 \\ &\leq \frac{4\Delta t^2}{ZN^n} \sum_j \frac{S(f_j^n)^2}{f_j^n} \Delta v \cdot \sum_j f_j^n \frac{(v_j - \widetilde{U}^n)^4}{4(T_e^n/m)^2} \Delta v \\ &\leq \Delta t^2 \sum_j \frac{S(f_j^n)^2}{f_j^n} \Delta v \cdot \frac{\max_k (v_k - \widetilde{U}^n)^4}{Z(T_e^n/m)^2}. \end{aligned}$$

Then

$$H^{n+1} \leq H^n + \Delta t \sum_j \left[S(f^n) \log\left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}}\right)_j + \Delta t (1 + \alpha^n) \frac{S(f_j^n)^2}{f_j^n}\right] \Delta v.$$

And, by applying Lemma A.4, we obtain

$$H^{n+1} \leq H^n + \Delta t \left[1 - \frac{4\Delta t \Omega^n T_e^n}{m \Delta v^2} \cdot \mathfrak{M}^n \frac{h_{\max}^n}{h_{\min}^n} (1 + \alpha^n)\right] \cdot \sum_j S(f^n)_j \log\left(\frac{f^n}{\mathbf{M}_{\widetilde{U}^n, T_e^n}}\right)_j \Delta v.$$

We conclude the proof by using the equality (2.14) and the inequality (2.16) which allow us to write that when $\Delta t < \Delta t_3^n$, we have $H^{n+1} \leq H^n$. To show that $H^n \geq H(f^\infty, T_e^\infty)$, we use the first point of Proposition 2.7, which does not depend on the time discretization.

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