Asymptotic Analysis 25 (2001) 93–107
IOS Press

The grazing collision limit for the Boltzmann–Lorentz model

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Abstract. The Lorentz operators are derived from either Boltzmann or Fokker–Planck collisions operators when considering a mixture of species with disparate masses [8]. The Fokker–Planck operator is the so called "grazing collision limit" of the Boltzmann operator as proved in [1,12,7]. In our simpler case, we improve the results by proving uniform in time convergence and by controling the speed of the trend to equilibrium. The results are based on a spectral analysis of the operators which share the same basis of eigenfunctions.

1. Introduction

The Fokker–Planck collision operator is usually considered as an approximation of the Boltzmann collision operator when the collisions become grazing. This has been proved in a series of papers, starting from Arsen'ev and Buryak who show in [1] this convergence result under restrictive assumptions on the scaterring cross section and on the initial data. A mathematical framework has also been given by Desvillettes for more physical situations, but which still exclude the Coulombian case; in [12], the scaterring cross-section is smooth and depends upon a small parameter ε which tends to zero. This parameter however does not have any precise physical meaning. A quite different asymptotics is used in [7] to treat the Coulombian case: here, the scattering cross section has a non integrable singularity when the relative velocity of the colliding particles tends to zero. Moreover, the small parameter involved in this asymptotics has an actual physical meaning: it is clearly identified to the plasma parameter. Recently, Villani [19] obtained a complete rigorous proof of this asymptotic problem in the space homogeneous situation, and for potentials which are not "too soft".

In this paper, we are concerned with a simplified collision operator which is known as the Boltzmann–Lorentz model in plasma physics. It is used to describe the effects of collisions of electrons with neutral particles. In first approximation, the electrons are assumed to diffuse with a stationary equilibrium distribution of target particles. The simplified Boltzmann–Lorentz model is then derived from the Boltzmann equation in the limit of small electron mass with respect to the mass of atoms; this asymptotics has been completely justified from the theory of kinetic collisional operators in [8]. The Boltzmann–Lorentz model can also be found in the framework of semi-conductors (see [4]). More sophisticated models of the same form have also been recently studied in the context of wave-particle modelling [10], cometary plasma [9], ionic plasma thurster. . .

In the same way, we can define a simplified Fokker–Planck–Lorentz model which can be derived from the Fokker–Planck–Landau equation in the limit of small electron mass with respect to the mass of ions (see again [8]). Physical situations actually exist for which this operator appears as the leading order collision term (see [14] for an example in the context of plasmas). From a probabilistic point of view, the Fokker–Planck–Lorentz operator describes a random walk of the particles on any sphere of constant energy.

In this paper, we are interested with the limit of the Boltzmann–Lorentz operator towards the Fokker–Planck–Lorentz model in the so-called grazing collision limit. Two situations are considered: a first one, denoted by "case 1" which corresponds to the asymptotics developed by Desvillettes in [12]; a second one, called "case 2", is the asymptotics introduced in [7] to treat the Coulombian case. In both cases, we show the convergence of the operators, but also of the solutions of the Cauchy problems associated with these operators: for the last, the convergence is uniform with respect to time, on any finite time interval. But, thanks to a precise spectral analysis, we can go further. First, we can show a uniform convergence result when time goes to infinity. Second, we have a precise knowledge of the convergence speed towards equilibrium. This spectral analysis also presents a great interest as a numerical point of view: it gives in particular exact solutions which allow the validation of numerical codes. Numerical experiments are the object of a forthcoming paper [6].

Our paper is organized in the following way. In part 2, we present the operators under consideration. We then introduce the grazing collision asymptotics and show that the Boltzmann–Lorentz operator tends towards the Fokker–Planck–Lorentz operator, for very regular distribution functions. Less regular situations are considered in part 3. A spectral analysis is then carried out. The key point relies on the fact that the two operators have the same eigenfunctions. Convergence results are then established for the eigenvalues associated with these eigenfunctions, either in case 1, or in the Coulombian case. The proofs are detailed in the three dimensional case, which is the real physical case; main results are also valid in two dimensions (see Remark 3.4). We show the convergence of the solutions of the Cauchy problem associated with the Boltzmann–Lorentz operator towards the solutions of the Cauchy problem associated with the Fokker–Planck–Lorentz operator. In part 4, we generalize this result in presence of an exterior magnetic field. The dependance with respect to the energy variable is also restored.

2. The Lorentz models

2.1. *Definition*

The "Boltzmann–Lorentz" model is the elastic-collision operator which has the following expression in d velocity-dimension [8]:

$$Q(f)(\omega) = \int_{S^{d-1}} B(\omega - \omega')\,[f(\omega') - f(\omega)]\,\mathrm{d}\omega', \tag{2.1}$$

where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$. The cross section B is a positive function which expression is directly connected to the type of interacting potential between the particles. More precisely, $B(\omega - \omega')$ only depends on $\|\omega - \omega'\|$.

Physical cases are mainly in three dimensions; from now on, we suppose that $d = 3$. Let us precise the notations. We introduce the following local orthonormal basis (e_1, e_2, ω). Using spherical coordinates, we can write

$$\omega' = \sin\theta(\cos\phi e_1 + \sin\phi e_2) + \cos\theta\omega, \tag{2.2}$$

where $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. As a physical point of view, the angle θ represents the so called “scaterring angle”, i.e., the angle of deviation undergone during a collision by a particle having ω as initial velocity and ω' as post-collisional velocity. The integrand $\mathrm{d}\omega'$ denotes the elementary area of S^2; with the above notations, we have $\mathrm{d}\omega' = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. Now, a simple computation shows that $\|\omega - \omega'\|^2 = 2(1 - \cos(\theta))$, so that $B(\omega - \omega')$ only depends on the scaterring angle θ (and not on ϕ), or, equivalently, on the scalar product $\omega \cdot \omega'$.

The Lorentz–Fokker–Planck operator is nothing but the classical Laplace–Beltrami operator on the unit sphere S^2. Still using spherical coordinates, it is defined by:

$$P(f) = \Delta_\omega f = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}\right]. \tag{2.3}$$

Let us recall that this operator in fact appears quite naturally, when considering for example collisions of heavy charged particles against light ones. We write the velocities v in spherical coordinates (i.e., $v = |v|\omega$). The leading order term of the Fokker–Planck–Landau collision operator is then precisely, up to a multiplicative function of the modulus of the velocity, the operator $P(f)$. We refer to [14] for a precise description of this.

2.2. *Cross sections for grazing collisions*

In the grazing collision asymptotics, the cross section B in the Boltzmann–Lorentz operator (2.1) is supposed to depend on a small parameter ε; we denote it by B^ε. The associated Boltzmann–Lorentz operator then writes

$$Q^\varepsilon(f)(\omega) = \int_{S^{d-1}} B^\varepsilon(\omega - \omega')\left[f(\omega') - f(\omega)\right]\mathrm{d}\omega'. \tag{2.4}$$

Let us now precise the definition of the cross section B^ε. As we have seen previously, the kernel B^ε only depends on one angle $\theta \in [0, \pi]$, so that we can write $B^\varepsilon(\omega - \omega')\,\mathrm{d}\omega' = \overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi$. Now for most potentials, $\overline{B}^\varepsilon$ is of the form [12]

$$\overline{B}^\varepsilon(\theta) = \frac{1}{\varepsilon^3}\overline{B}\left(\frac{\theta}{\varepsilon}\right), \tag{2.5}$$

with $\overline{B}(\theta) = \sigma(\theta)\sin(\theta)\chi_{[0,\pi]}(\theta)$; we denote by $\chi_{[a,b]}$ the characteristic function of the interval $[a, b]$ and σ is a positive function. We then have:

$$\overline{B}^\varepsilon(\theta) = \frac{1}{\varepsilon^3}\sigma\left(\frac{\theta}{\varepsilon}\right)\sin\left(\frac{\theta}{\varepsilon}\right)\chi_{[0,\varepsilon\pi]}(\theta). \tag{2.6}$$

Unfortunately, this asymptotics does not allow to treat the Coulombian case, which is the most relevant physical case. In the last, the kernel writes

$$\overline{B}^{\varepsilon}(\theta) = \sigma(\theta)\frac{1}{\mathrm{Log}(1/\sin\frac{\varepsilon}{2})}\,\frac{\sin\theta}{[\sin\frac{\theta}{2}]^4}\,\chi_{[\varepsilon,\pi]}(\theta), \tag{2.7}$$

where the positive function σ is such that $\sigma(0) \neq 0$, and the parameter ε has a physical meaning: it is what physicists call the "plasma parameter" [7]. The logarithm factor represents the so called "Coulombian logarithm".

From now on, we denote by "case 1" the non-Coulombian case which corresponds to cross-sections of the form (2.6), and by "case 2" the Coulombian one; in the last, the cross sections are given by (2.7).

2.3. *The grazing limit for smooth distribution functions*

In this part, we present a formal justification of the so called grazing collision limit, which can be justified in the context of regular distribution functions. In the next paragraph, we shall perform a spectral analysis of the operators Q^{ε} and P. This will allow to get a more precise result, for less regular distribution functions; but it will also give a precise result concerning the trend to equilibrium.

Proposition 2.1. *Let $f \in C^3(S^2)$. We consider the Boltzmann–Lorentz operator* (2.4) *with cross sections of the form* (2.6)*, i.e., case* 1*. We suppose moreover that the cross section σ is such that*:

$$C = \int_0^{\pi} \mu^2\sigma(\mu)\sin\mu\,\mathrm{d}\mu < +\infty. \tag{2.8}$$

Then, for all $\omega \in S^2$, we have:

$$\lim_{\varepsilon\to 0} Q^{\varepsilon}(f)(\omega) = C\frac{\pi}{2}\,\Delta_{\omega} f(\omega).$$

Proof. The function f, defined on the sphere, is first extended on the whole space in such a way that the third derivative of f remains bounded. For example, one can use the following extension of f: $\tilde{f}(\omega) = f(\omega/\|\omega\|)\Psi(\|\omega\|)$, where $\omega \in \mathbb{R}^d$ and Ψ is a C^3 function which is equal to 1 in a neighborhood of 1. We then use the following Taylor expansion

$$f(\omega') = f(\omega) + (\omega' - \omega)_i\partial_i\tilde{f}(\omega) + \frac{(\omega' - \omega)^2_{i,j}}{2}.\partial^2_{i,j}\tilde{f}(\omega) + R(\omega,\omega'),$$

where the integral remainder is such that $|R(\omega,\omega')| \leqslant M\|\omega - \omega'\|^3$, due to the regularity assumptions on f. Note that now ω are vectors in $\mathbb{R}^3$ and ∂_i denote the partial derivatives of f in the ith direction and $x_{i,j}$ is the tensor product $x_i x_j$. The index i and j varie in $\{1,2,3\}$ and we use the Einstein convention (summation of repeated index).

We write ω' in the basis (2.2) linked to ω. Recall that in this basis $\omega' - \omega = \sin\theta(\cos\phi e_1 + \sin\phi e_2) + (\cos\theta - 1)\omega$, since $e_3 = \omega$. The first term in (2.4) corresponding to derivatives of order 1 vanishes using the eveness of the kernel. It remains:

$$Q^{\varepsilon}(f)(\omega) = \frac{1}{2}\partial^2_{i,j}\tilde{f}(\omega)\int_{S^2}\overline{B}^{\varepsilon}(\theta)\big(\omega' - \omega\big)_{i,j}\,\mathrm{d}\theta\,\mathrm{d}\phi + R^{\varepsilon}(\omega),$$

where θ, ϕ are the spherical coordinate of ω' in the basis (2.2), while R^ε is defined by: $R^\varepsilon(\omega) = \int_{S^2} \overline{B}^\varepsilon(\theta) R(\omega - \omega')\,\mathrm{d}\theta\,\mathrm{d}\phi$. Let us now consider the first integral term, i.e., the matrix defined by:

$$\Phi^\varepsilon_{i,j}(\omega) \stackrel{\text{def}}{=} \int_{S^2} \overline{B}^\varepsilon(\theta)\big(\omega' - \omega\big)_{i,j}\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

This matrix is diagonal since it vanishes for each $i \neq j$ by eveness. Then, for $i = j = 3$, we have, integrating in ϕ and using (2.6)

$$\Phi^\varepsilon_{3,3}(\omega) = 2\pi \int_0^\pi \big(\cos\theta - 1\big)^2 \overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta = \frac{8\pi}{\varepsilon^2}\int_0^\pi \sin^4\frac{\varepsilon\mu}{2}\sigma(\mu)\sin\mu\,\mathrm{d}\mu \to 0, \quad \text{as } \varepsilon \to 0,$$

thanks to the assumption (2.8). By integrating over ϕ, we also have

$$\Phi^\varepsilon_{1,1}(\omega) = \Phi^\varepsilon_{2,2}(\omega) = \pi \int_0^\pi \sin^2(\theta)\overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta = \frac{\pi}{\varepsilon^2}\int_0^\pi \sin^2(\varepsilon\mu)\sigma(\mu)\sin\mu\,\mathrm{d}\mu \to \pi C, \quad \text{as } \varepsilon \to 0,$$

by using the assumption (2.8) and the definition of C. Therefore, $\Phi^\varepsilon \to \Phi$, when $\varepsilon \to 0$, where Φ is the projection matrix onto the plane perpendicular to ω multiplied by πC, that is to say $\Phi(\omega) = \pi C(\mathrm{Id} - (\omega \times \omega)/\|\omega\|^2)$. Moreover, we have: $\partial_{i,j}\tilde{f}(\omega) \cdot \Phi_{i,j}(\omega) = \nabla \cdot (\Phi(\omega)\nabla\tilde{f}) = \Delta_\omega \tilde{f}(\omega)$, for $\omega \in S^2$. Using the following upper bound $\|\omega' - \omega\|^3 \leqslant 8\sin^3(\theta/2)$, we get an estimate for the remainder term

$$|R^\varepsilon(\omega)| \leqslant \frac{16\pi M}{\varepsilon^2}\int_0^\pi \sin^3\frac{\varepsilon\mu}{2}\sigma(\mu)\sin\mu\,\mathrm{d}\mu,$$

which shows that this term vanishes when $\varepsilon \to 0$. This end the proof. □

Using the semigroup theory, we deduce from Proposition 2.1 that T^ε, the continuous semigroup associated with Q^ε, converges towards the semigroup T associated with the Laplace–Beltrami operator. Then, the Pazy theorem [16] allows to obtain for any arbitrary large time $T > 0$ and any initial data $f_0 \in L^p$, the following estimate:

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\big\|T^\varepsilon(t)(f_0) - T(t)(f_0)\big\|_{L^p(S^2)} = 0.$$

The convergence is thus uniform in time but on bounded intervals of the type $[0,T]$. This also implies convergence results in some $L^\infty(0,\infty,L^p(S^2))$ weak (like in the work of Desvillettes). We shall not detail the proof of this result since it does not give us informations on the trend to equilibrium, i.e., on the large time behaviour of the solution. It is however expected that, for large time, the solution homogeneizes, i.e., tends towards a constant state (the average of the initial distribution function on the unit sphere). But, it is not clear, using such techniques, if the convergence is uniform in time; the trend towards this constant state can be as slow as the grazing collision parameter ε tends to 0. In the next part, we shall answer to this point using spectral analysis of the operator which gives more precise (explicit, uniform in time) result on the existence of solution and on its grazing collision limit.

3. Spectral analysis of the Boltzmann–Lorentz model

3.1. The spectral analysis

The spherical harmonics Y_{lm} defined by [15]

$$Y_{lm}(\theta,\phi)=(-1)^m\mathrm{i}^l\left[\frac{l+1/2}{2\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2}\mathrm{e}^{\mathrm{i}m\phi}(\sin\theta)^m\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m P_l(\cos\theta),$$

where P_l denotes the Legendre polynomials, are eigenfunctions of the Laplace–Beltrami operator. More precisely, we have, for $l\geqslant 0$ and $-l\leqslant m\leqslant l$ [15]:

$$-\Delta_\omega Y_{lm}=\nu_l Y_{lm},\quad \text{with } \nu_l=l(l+1).$$

These functions form an orthonormal basis of the space $L^2(S^2)$. Moreover, they are also eigenfunctions of the operator Q^ε [3,11], and we have:

$$-Q^\varepsilon(Y_{lm})=\nu_l^\varepsilon Y_{lm},\quad \text{with } \nu_l^\varepsilon=2\pi\int_0^\pi\big[1-P_l\big(\cos\theta\big)\big]\overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta. \tag{3.1}$$

These eigenvalues first satisfy $\nu_l^\varepsilon\geqslant 0$, $\nu_0^\varepsilon=0$, as a direct consequence of the fact that [13]: $\forall x\in[-1,1]$, $-1\leqslant P_l(x)\leqslant 1, P_0(x)=1$. In order to get a uniform bound, in terms of ε, of the eigenvalues, we first show the following preparatory lemma:

Lemma 3.1. *Let us suppose that*:

$$\int_0^\pi\sigma(\theta)\sin\theta\theta^2\,\mathrm{d}\theta<+\infty,\quad \textit{in case } 1, \tag{3.2}$$

$$\sup_{\theta\in[0,\pi]}\sigma(\theta)<+\infty,\quad \textit{in case } 2. \tag{3.3}$$

Then, we have:

$$\exists C_1(\sigma)>0 \textit{ such that }\quad \forall\varepsilon>0,\int_0^\pi\theta^2\overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta\leqslant C_1(\sigma). \tag{3.4}$$

Moreover, there exists a positive constant $C(\sigma)$, independent of ε, such that:

$$\textit{for all } l\geqslant 0,\quad \nu_l^\varepsilon\leqslant C(\sigma)\nu_l. \tag{3.5}$$

Proof. We first prove (3.4). The first case results from a simple change of variables. In the Coulombian case, we have:

$$\int_0^\pi\theta^2\overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta=\frac{2}{\mathrm{Log}(1/\sin\frac{\varepsilon}{2})}\int_\varepsilon^\pi\theta^2\sigma(\theta)\frac{\cos\frac{\theta}{2}}{[\sin\frac{\theta}{2}]^3}\,\mathrm{d}\theta\leqslant 16\sup_{[0,1]}\left(\frac{\arcsin x}{x}\right)^2\sup_{\theta\in[0,\pi]}\sigma(\theta),$$

which gives the expected estimate. Let us now show (3.5). Since [13] $P_l(1) = 1,\ \sup_{t\in[-1,1]} |P_l'(t)| \leqslant l(l+1)/2$, we have:

$$|1 - P_l(1+x)| = \left| \int_0^x P_l'(1+t)\,\mathrm{d}t \right| \leqslant |x| \frac{l(l+1)}{2}.$$

Applying this to $x = \cos(\theta) - 1$, we get, since $|x| \leqslant \theta^2/2$:

$$\nu_l^\varepsilon \leqslant \pi \frac{l(l+1)}{2} \int_0^\pi \theta^2 \overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta,$$

which gives (3.5) for $C(\sigma) = (\pi/2)C_1(\sigma)$.

3.2. *The Cauchy problem*

We are interested with the following Cauchy problem

$$\frac{\partial f^\varepsilon}{\partial t} = Q^\varepsilon(f^\varepsilon), \quad f^\varepsilon(t=0) = f_0, \tag{3.6}$$

where f_0 is a given function in $L^2(S^2)$. We split this function in the orthonormal basis formed by the spherical harmonics, i.e., we write:

$$f_0 = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm} Y_{lm}, \quad \text{with } \|f_0\|^2_{L^2(S^2)} = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm}^2 < +\infty. \tag{3.7}$$

From the above spectral analysis, we easily get:

Proposition 3.2. *The function f^ε defined by*

$$f^\varepsilon(t) = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm} \exp(-\nu_l^\varepsilon t) Y_{lm} \tag{3.8}$$

is a weak solution, in the space $L^\infty((0,\infty); L^2(S^2))$, of (3.6). *Moreover, if the initial data is such that*

$$\sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm}^2 \nu_l < +\infty, \tag{3.9}$$

then the above solution is the unique one such that $f^\varepsilon \in L^\infty(0,\infty; H^1(S^2))$ and $Q^\varepsilon(f^\varepsilon) \in L^\infty(0,\infty; H^{-1}(S^2))$.

Proof. The first point results from (3.1) and the non negativity of the ν_l^ε. Now condition (3.9) implies that $Q^\varepsilon(f^\varepsilon) \in L^\infty((0,\infty); H^{-1}(S^2))$. In fact any $\varphi \in H^1(S^2)$ writes

$$\varphi = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} b_{lm} Y_{lm}, \quad \text{with } \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} b_{lm}^2 \nu_l < +\infty,$$

so that by Cauchy–Schwartz inequality, we have, for any fixed $L > 0$,

$$\left| \int_{S^2} \left(\sum_{l \leqslant L} \sum_{m=-l}^{m=l} a_{lm} \nu_l^\varepsilon Y_{lm} \right) \varphi \, \mathrm{d}\omega \right| = \left| \sum_{l \leqslant L} \sum_{m=-l}^{m=l} a_{lm} b_{lm} \nu_l^\varepsilon \right|$$
$$\leqslant C(\sigma) \sqrt{\sum_{l \geqslant 0} \sum_{m=-l}^{m=l} a_{lm}^2 \nu_l} \sqrt{\sum_{l \geqslant 0} \sum_{m=-l}^{m=l} b_{lm}^2 \nu_l},$$

thanks to the estimate (3.5). The Banach–Steinhaus theorem shows that $\sum_{l \geqslant 0} \sum_{m=-l}^{m=l} a_{lm} \nu_l^\varepsilon Y_{lm} \in H^{-1}(S^2)$, so that $Q^\varepsilon(f^\varepsilon) \in L^\infty((0,\infty); H^{-1}(S^2))$. We also have, for all $t > 0$,

$$\langle Q^\varepsilon(f^\varepsilon)(t), f^\varepsilon(t) \rangle_{H^{-1},H^1} = -\sum_{l \geqslant 0} \sum_{m=-l}^{m=l} a_{lm}^2 \nu_l^\varepsilon \exp(-2\nu_l^\varepsilon t) \leqslant 0,$$

which gives

$$\int_{S^2} \left(f^\varepsilon(t,\omega) \right)^2 \mathrm{d}\omega \leqslant \int_{S^2} \left(f_0(\omega) \right)^2 \mathrm{d}\omega,$$

and shows the uniqueness. □

In the same way, the function f defined by

$$f(t) = \sum_{l \geqslant 0} \sum_{m=-l}^{m=l} a_{lm} \exp(-\nu_l t) Y_{lm} \tag{3.10}$$

is a weak solution, in the space $L^\infty((0,\infty); L^2(S^2))$ of the Cauchy problem:

$$\frac{\partial f}{\partial t} = \Delta_\omega f, \quad f(t=0) = f_0. \tag{3.11}$$

Moreover, if the initial condition satisfies (3.9), then the above solution is the unique one such that $f \in L^\infty((0,\infty); H^1(S^2))$ and $\Delta_\omega f \in L^\infty((0,\infty); H^{-1}(S^2))$.

3.3. *The grazing collision limit*

Let us now examine the convergence of the eigenvalues when $\varepsilon \to 0$.

Lemma 3.3. *Let us suppose that*:

$$\int_0^\pi \sigma\theta \sin\theta\theta^2 \, \mathrm{d}\theta = \frac{2}{\pi}, \quad \textit{in case } 1, \tag{3.12}$$

$$\sigma(0) = \frac{1}{2\pi}, \quad \textit{in case } 2. \tag{3.13}$$

Let $\varphi = \varphi(\theta)$ *be any function of class* C^2 *such that* $\varphi(0) = \varphi'(0) = 0$. *Then*:

$$\int_{S^2} B^\varepsilon(\omega)\varphi(\theta)\,\mathrm{d}\omega \to 2\varphi''(0), \quad \textit{when } \varepsilon \to 0. \tag{3.14}$$

Proof. In both cases, we use a change of variables followed by a Taylor expansion around point 0. We get in the first case:

$$\int_{S^2} B^\varepsilon(\omega)\varphi(\omega)\,\mathrm{d}\omega \to \pi\varphi''(0)\left(\int_0^\pi \sigma x \sin x x^2\,\mathrm{d}x\right), \quad \text{when } \varepsilon \to 0.$$

In the Coulombian case, we have:

$$\int_{S^2} B^\varepsilon(\omega)\varphi(\omega)\,\mathrm{d}\omega = \frac{4\pi}{\mathrm{Log}(1/\sin\frac{\varepsilon}{2})}\int_\varepsilon^\pi (\varphi\sigma)(\theta)\frac{\cos(\frac{\theta}{2})}{[\sin\frac{\theta}{2}]^3}\,\mathrm{d}\theta \to 4\pi\sigma(0)\varphi''(0), \quad \text{when } \varepsilon \to 0,$$

because $\varphi(0) = \varphi'(0) = 0$.

We deduce from this the:

Proposition 3.4. *Under the hypotheses of Lemma* 3.3, *we have*:

$$\textit{for all } l \geqslant 0, \quad \nu_l^\varepsilon \to \nu_l = l(l+1), \quad \textit{when } \varepsilon \to 0. \tag{3.15}$$

For any finite time interval $[0,T]$, *we have*:

$$\sup_{[0,T]} \|f^\varepsilon(t) - f(t)\|_{L^2(S^2)} \to 0, \quad \textit{when } \varepsilon \to 0. \tag{3.16}$$

Proof. We first apply (3.14) with $\varphi(\theta) = 1 - P_l(\cos\theta)$. The point (3.15) is then a simple consequence of the fact that: $\varphi''(0) = P_l'(1) = l(l+1)/2$. We deduce from (3.8) and (3.10) that

$$f^\varepsilon(t) - f(t) = \sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}Y_{lm}\left[\exp\left(-\nu_l^\varepsilon t\right) - \exp(-\nu_l t)\right], \tag{3.17}$$

with $\sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}^2 < +\infty$, so that we can write, for any $t \in [0,T]$:

$$\|f^\varepsilon(t) - f(t)\|_{L^2(S^2)}^2 \leqslant 2\sum_{l\geqslant L}\sum_{m=-l}^{m=l} a_{lm}^2 + \sum_{l=0}^{L-1}\sum_{m=-l}^{m=l} a_{lm}^2|\nu_l^\varepsilon - \nu_l|T.$$

The first sum is the remainder of a convergent series, which can be arbitrary small for a sufficiently large index L. The second part is then a finite sum of terms which all vanish when $\varepsilon \to 0$, which shows (3.16) and ends the proof. □

3.4. *The large time behaviour*

We are now interested with the large time behaviour of the solutions (3.8) and (3.10). In order to do this, we need a positive lower bound for the opposite of the eigenvalues ν_l^ε and ν_l, which means we have to "eliminate" the zero eigenvalue which corresponds to the equilibrium states. Since the kernels of the Lorentz operators are formed of constant functions, we easily get following result (we state it here for Q^ε, but this result also trivially holds for P):

Lemma 3.5. *Let f^ε by given by* (3.8). *The function g^ε defined by $g^\varepsilon = f^\varepsilon - I(f_0)$, with $I(f_0) = \frac{1}{4\pi}\int_{S^2} f_0\,\mathrm{d}\omega$, is of zero mean value over the unit sphere and it satisfies the following Cauchy problem*:

$$\frac{\partial g^\varepsilon}{\partial t} = Q^\varepsilon(g^\varepsilon), \quad g^\varepsilon(t=0) = g_0, \quad \textit{with } g_0 = f_0 - I(f_0). \tag{3.18}$$

Moreover, using the notation (3.7), *we have*:

$$g_0 = \sum_{l>0}\sum_{m=-l}^{m=l} a_{lm}Y_{lm}, \quad g^\varepsilon(t) = \sum_{l>0}\sum_{m=-l}^{m=l} a_{lm}\exp(-\nu_l^\varepsilon t)Y_{lm}. \tag{3.19}$$

We now analyze the convergence towards equilibrium. First, since $g = f - I(f_0)$, with f given by (3.10), is of zero mean value, we have the following ellipticity relation, where λ_1 denotes the first non zero eigenvalue of the Laplace–Beltrami operator [2]:

$$\|\nabla_\omega g\|^2_{L^2(S^2)} \geqslant \frac{1}{\lambda_1}\|g\|^2_{L^2(S^2)};$$

this gives $\int_{S^2} g^2\,\mathrm{d}\omega \leqslant \exp(-\frac{1}{\lambda_1}t)\int_{S^2} g_0^2\,\mathrm{d}\omega$, which shows that g converges exponentially fast towards zero, when time goes to infinity. Now, if there exists a positive constant C such that, for all ε we have $B^\varepsilon \geqslant C$, then the same behaviour holds for g^ε. Our aim is to generalize this result for situations where the above assumption on the kernel is not satisfied. In order to do so, we need a uniform lower bound for the ν_l^ε.

Lemma 3.6. *We suppose the assumption* (3.2) *fulfilled, with σ non identically equal to zero. In the Coulombian case, we suppose that there exists positive constants θ_0 and σ_0, $\theta_0 < \pi/2$, such that*: $\inf_{\theta\in[0,\theta_0]}\sigma(\theta) \geqslant \sigma_0$. *Then, we have*:

$$\exists\varepsilon_0 > 0,\ C_2(\sigma) > 0 \textit{ such that } \forall\varepsilon \in\,]0,\varepsilon_0[\,, \quad \int_0^{\frac{\pi}{2}} \theta^2\overline{B}^\varepsilon(\theta)\,\mathrm{d}\theta \geqslant C_2(\sigma). \tag{3.20}$$

Moreover, there exists positive constants ν^0 and ε_0, such that:

$$\textit{for all } l \geqslant 1 \textit{ and all } \varepsilon \in [0,\varepsilon_0], \quad \nu_l^\varepsilon \geqslant \nu^0. \tag{3.21}$$

Proof. We first prove (3.20). Like in Lemma 3.1, the first case results from a simple change of variables; we find $\varepsilon_0 = 1/2$ and $C_2(\sigma) = \int_0^\pi \sigma(u) \sin u u^2 \, du$. In the Coulombian case, we have, choosing $\varepsilon_0 = \theta_0$,

$$\int_0^{\frac{\pi}{2}} \theta^2 \overline{B}^\varepsilon(\theta) \, d\theta \geqslant \frac{8\sqrt{2}}{\mathrm{Log}(1/\sin\frac{\varepsilon}{2})} \sigma_0 \, \mathrm{Log}\left(\frac{\theta_0}{\varepsilon}\right),$$

which gives the following estimate for ε_0 sufficently small: $\int_0^{\frac{\pi}{2}} \theta^2 \overline{B}^\varepsilon(\theta) \, d\theta \geqslant 4\sqrt{2}\sigma_0$. Let us now get a uniform lowerbound for ν_l^ε. We recall the Laplace formula [13] ($\mathrm{i}^2 = -1$)

$$P_l(x) = \frac{1}{\pi} \int_0^\pi \left(x + \mathrm{i}\sqrt{1-x^2} \cos\phi\right)^l d\phi, \quad -1 \leqslant x \leqslant 1,$$

which gives: $|P_l(x)| \leqslant \frac{1}{\pi} \int_0^\pi (x^2 + (1-x^2)\cos^2\phi)^{l/2} \, d\phi$. The quantity inside the integral being less than 1, we deduce that, for any $l \geqslant 2$, we have $|P_l(x)| \leqslant \frac{1}{\pi} \int_0^\pi (x^2 + (1-x^2)\cos^2\phi) \, d\phi$, so that: $|P_l(x)| \leqslant (1+x^2)/2$. Now this last relation is also valid for $l = 1$, since $P_1(x) = x$, which finally gives: $1 - P_l(x) \geqslant (1-x^2)/2$, for $l \geqslant 1$. Using the inequality $\sin(\theta) \geqslant (2/\pi)\theta$, for $\theta \in [0, \pi/2]$, we deduce the following lower bound, for any $l \geqslant 1$:

$$\nu_l^\varepsilon \geqslant \pi \int_0^\pi \sin^2\theta \overline{B}^\varepsilon(\theta) \, d\theta \geqslant \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \theta^2 \overline{B}^\varepsilon(\theta) \, d\theta,$$

which gives the expected result, once condition (3.20) is fulfilled.

We can now derive the asymptotic behaviour when time goes to infinity.

Theorem 3.7. *There exists a positive constant* ε_0 *such that*:

$$\sup_{\varepsilon \in [0,\varepsilon_0]} \|f^\varepsilon(t) - I(f_0)\|_{L^2(S^2)} \to 0, \quad \textit{when } t \to +\infty. \tag{3.22}$$

More precisely, using the notations of Lemma 3.6, *we have*:

$$\sup_{\varepsilon \in [0,\varepsilon_0]} \|f^\varepsilon(t) - I(f_0)\|_{L^2(\mathbb{R}^3 \times S^2)} \leqslant \exp\left(-\nu^0 t\right) \|f_0\|_{L^2(\mathbb{R}^3 \times S^2)}. \tag{3.23}$$

Proof. In order to prove the uniform convergence result (3.22), we in fact only need a weaker form of the estimate given in Lemma 3.6. We here suppose that:

$$\text{for all } l \geqslant 1, \ \exists \nu_l^0 > 0, \ \varepsilon_l^0 > 0 \text{ such that for all } \varepsilon \in \left[0, \varepsilon_l^0\right] \quad \nu_l^\varepsilon \geqslant \nu_l^0. \tag{3.24}$$

Now by (3.19), we have, for all $t > 0$,

$$\|g^\varepsilon(t)\|^2_{L^2(S^2)} \leqslant \sum_{0<l\leqslant L-1} \sum_{m=-l}^{m=l} a_{lm}^2 \exp\left(-2\nu_l^\varepsilon t\right) + \sum_{l\geqslant L} \sum_{m=-l}^{m=l} a_{lm}^2,$$

because the ν_l^ε are all non negative. Now, since $g_0 \in L^2(S^2)$, the series $\sum_{l>0}\sum_{m=-l}^{m=l} a_{lm}^2$ is convergent, so that the second sum in the above expression (which is independent of ε) can be arbitrary small for L large enough. On the other hand, the first term is a finite sum of terms that all converge exponentially fast towards 0 when time goes to infinity, on account of the above assumption $\nu_l^\varepsilon \geqslant \nu_l^0 > 0$. More precisely, there exists a positive constant ε_0 such that, for all $\varepsilon \in [0, \varepsilon_0]$, we have:

$$\sum_{0<l\leqslant L-1}\sum_{m=-l}^{m=l} a_{lm}^2 \exp\bigl(-2\nu_l^\varepsilon t\bigr) \leqslant \sum_{0<l\leqslant L-1}\sum_{m=-l}^{m=l} a_{lm}^2 \exp\bigl(-2\nu_l^0 t\bigr).$$

When t goes to infinity, this sum tends then to zero uniformaly with respect to ε, which finally gives (3.22).

In the three dimensional case (which is the case under consideration here), we have a more precise result. In fact, thanks to the uniform lower bound (3.21), we get the estimate (3.23): the decrease, when time goes to infinity, is thus exponential.

Remark 1. In the two dimensional case, we obtain the same uniform convergence result (3.22). In case 1, the decrease, when time goes to infinity, is exponential, because we can find for the opposite of the eigenvalues a uniform lower bound with respect to ε, such as in Lemma 3.6. But in the Coulombian case, we could only manage to find the weaker estimate (3.24). □

4. Some complements

4.1. The dependence with respect to the modulus of the velocity variable

The distribution function in fact depends on the whole velocity variable $v = \rho\omega$, $\rho = |v|$, although the differential operator only acts on the variable $\omega \in S^2$. If we recover the whole velocity variable, the Cauchy problems (3.6) and (3.11) respectively write [14]

$$\frac{\partial f^\varepsilon}{\partial t} = A^\varepsilon\bigl(f^\varepsilon\bigr) = |v|^\gamma Q^\varepsilon\bigl(f^\varepsilon\bigr), \quad f^\varepsilon(t=0) = f_0, \tag{4.1}$$

$$\frac{\partial f}{\partial t} = A(f) = |v|^\gamma \Delta_\omega f, \quad f(t=0) = f_0. \tag{4.2}$$

The power γ is directly connected to the type of intercating potential between the particles. The case $\gamma > 0$ corresponds to what is usually called “hard” potentials, $\gamma < 0$ to “soft” potentials, while $\gamma = 0$ is the particular case of Maxwellians molecules.

The unknown is $f = f(t, \rho, \omega)$, with $\rho \in \mathbb{R}^+$, $\omega \in S^2$. We denote by $L_W^2(0, \infty)$ and X the following weighted spaces

$$L_W^2(0,\infty) = \left\{\psi = \psi(\rho), \int_0^{+\infty} \psi(\rho)\rho^2\,\mathrm{d}\rho < +\infty\right\},$$

$$X = L_W^2\bigl((0,\infty); L^2\bigl(S^2\bigr)\bigr) = \left\{\psi = \psi(\rho,\omega), \int_{S^2}\int_0^{+\infty} \psi^2(\rho,\omega)\rho^2\,\mathrm{d}\rho\,\mathrm{d}\omega < +\infty\right\}.$$

We split the square integrable initial data $f_0 = f_0(\rho,\omega)$ ($\rho = |v|$) in the following way

$$f_0(\rho,\omega) = \sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}(\rho)Y_{lm}(\omega),$$

with $\|f_0\|_X^2 = \int_0^\infty \sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}^2(\rho)\rho^2\,\mathrm{d}\rho < +\infty$. With a proof similar to that of Proposition 3.2, we easily get:

Proposition 4.1. *The function f^ε defined by*

$$f^\varepsilon(t,\rho,\omega) = \sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}(\rho)\exp\bigl(-\nu_l^\varepsilon\rho^\gamma t\bigr)Y_{lm}(\omega) \tag{4.3}$$

is a weak solution, in the space $L^\infty(0,\infty;X)$, of (4.1). *Moreover, if the initial data is such that*

$$\int_{\mathbb{R}^3}\sum_{l\geqslant 0}\sum_{m=-l}^{m=l} a_{lm}^2(\rho)\nu_l\rho^2\,\mathrm{d}\rho < +\infty,$$

then the above solution is the unique one such that

$$f^\varepsilon \in L^\infty\bigl(0,\infty;L_W^2\bigl((0,\infty);H^1\bigl(S^2\bigr)\bigr)\bigr) \quad \textit{and} \quad Q^\varepsilon\bigl(f^\varepsilon\bigr) \in L^\infty\bigl(0,\infty;L_W^2\bigl((0,\infty);H^{-1}\bigl(S^2\bigr)\bigr)\bigr).$$

Concerning the large time behaviour, we will not obtain in general (i.e., for any type of potentials) an exponential decrease when time goes to infinity. With the notations of Lemma 3.5, the function g^ε defined by

$$g^\varepsilon(t,x,\omega) = \sum_{l>0}\sum_{m=-l}^{m=l} a_{lm}(\rho)\exp\bigl(-\nu_l^\varepsilon\rho^\gamma t\bigr)Y_{lm}(\omega)$$

satisfies the following Cauchy problem $\partial g^\varepsilon/\partial t = A^\varepsilon(g^\varepsilon)$, $g^\varepsilon(t=0) = f_0 - I(f_0)$. Thanks to the estimate (3.21), we have, for ε small enough,

$$\|g^\varepsilon(t)\|_{L^2(\mathbb{R}^3\times S^2)}^2 \leqslant \int_{\mathbb{R}^3}\sum_{l>0}\sum_{m=-l}^{m=l} a_{lm}^2(\rho)\exp\bigl(-2\nu^0\rho^\gamma t\bigr)\rho^2\,\mathrm{d}\rho$$

so that this quantity tends to zero when time goes to infinity, but it tends exponentially fast towards zero only in the case $\gamma = 0$. For $\gamma > 0$, this exponential decrease holds outside any ball centered at zero with arbitrary small radius, while for $\gamma < 0$, it happens inside any ball centered at zero with arbitrary large radius. Let us introduce the ρ dependent function defined by: $I(f_0)(\rho) = \frac{1}{4\pi}\int_{S^2} f_0(\rho,\omega)\,\mathrm{d}\omega$. Gathering all the cases, we have shown the following result:

Theorem 4.2. *There exists a positive constant ε_0 such that*:

$$\sup_{\varepsilon\in[0,\varepsilon_0]}\bigl\|f^\varepsilon(t) - I(f_0)\bigr\|_X \to 0, \quad \textit{when } t \to +\infty. \tag{4.4}$$

4.2. The presence of an external magnetic field

The action of a magnetic field B on the spherical harmonics is well known [11]. In particular, when we take the direction of the vector B as the axis of the spherical coordinates, one has the following identity, for any $\omega \in S^2$: $(\omega \wedge B) \cdot \nabla_\omega Y_{lm}(\omega) = -\mathrm{i}m\|B\| Y_{lm}(\omega)$. This allows an explicit computation of the solution of the following Cauchy problem

$$\frac{\partial f^\varepsilon}{\partial t} + (\omega \wedge B) \cdot \nabla_\omega f^\varepsilon = Q^\varepsilon(f^\varepsilon), \quad f^\varepsilon(t=0) = f_0, \tag{4.5}$$

with f_0 in $L^2(S^2)$. In fact, still using the expansion (3.7) of f_0, we have:

Proposition 4.3. *Let $\nu^\varepsilon_{l,m} = \nu^\varepsilon_l - \mathrm{i}m\|B\|$. The function f^ε defined by*

$$f^\varepsilon(t) = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm} \exp\bigl(-\nu^\varepsilon_{l,m} t\bigr) Y_{lm} \tag{4.6}$$

is a weak solution, in the space $L^\infty((0,\infty); L^2(S^2))$, of (4.5). *Moreover, if the initial data is such that* (3.9) *holds, then the above solution is the unique one such that*

$$f^\varepsilon \in L^\infty\bigl((0,\infty); H^1\bigl(S^2\bigr)\bigr) \quad \textit{and} \quad Q^\varepsilon\bigl(f^\varepsilon\bigr) \in L^\infty\bigl((0,\infty); H^{-1}\bigl(S^2\bigr)\bigr).$$

In the same way, the function f defined by

$$f(t) = \sum_{l\geqslant 0} \sum_{m=-l}^{m=l} a_{lm} \exp(-\nu_{l,m} t) Y_{lm},$$

with $\nu_{l,m} = \nu_l - \mathrm{i}m\|B\|$ satisfies the Cauchy problem

$$\frac{\partial f}{\partial t} + (\omega \wedge B) \cdot \nabla_\omega f = \Delta_\omega f, \quad f(t=0) = f_0, \tag{4.7}$$

with the same data f_0. Finally, we also keep the following exponential decay when time goes to infinity:

$$\sup_{\varepsilon\in[0,\varepsilon_0]} \bigl\|f^\varepsilon(t) - I(f_0)\bigr\|_{L^2(\mathbb{R}^3\times S^2)} \leqslant \exp\bigl(-\nu^0 t\bigr) \|f_0\|_{L^2(\mathbb{R}^3\times S^2)}. \tag{4.8}$$

Acknowledgements

The authors acknowledge support from the GdR SPARCH of the Centre National de la Recherche Scientifique (France) and from the TMR project “Asymptotic methods in kinetic theory” (TMR number: ERB FMRX CT97 0157), run by the European Community. They also would like to thank the referee for helpfull remarks.

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