Mathematical Models and Methods in Applied Sciences

MULTIFLUID IONIZATION MODELS

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In this paper, we present a multi-fluid ionization model. We prove that this stationary, mono-dimensional model has a maximal solution which is not global at variance with the mono-species case and we present a numerical method for solving this highly singular system of ordinary differential equations. Numerical results are compared with those obtained for other models.

Keywords: Singular system of ordinary differential equations, Cauchy problem, Fluid models, Numerical schemes, Ionization.

1. Introduction

In many problems encountered in plasma physics, ionization processes play an essential role. This is the case when considering the extraction of an ion beam from a neutral plasma.

In a reaction chamber, electrons ionize a gas through fairly complicated ionization processes: the ion beam is extracted from the generated plasma through a small aperture facing an electrode carried to an electric potential which is strongly negative with respect to the plasma potential. Indeed, simple ionization models are able to predict the density and the current density of the ion beam at the extraction. Moreover, these quantities, which can be computed analytically, are independent of the ionization process which implies that one does not need to model the plasma from which the ion beam is extracted. For these results, we refer for instance to [10].

These simple models suppose that only a single species of ions exists in the plasma. This hypothesis is not realistic in the applications : a real plasma contains several species of ions. The purpose of this paper is to study a multispecies ionization fluid model and to determine the densities and current densities at the extraction of the beam. Although an analytic approach is no longer available in the multispecies case, we will be able to give precise qualitative answers: the densities and the current densities of each ion species at the extraction now depend on the ionization process but we indicate a simple way of computing them numerically.

The paper is organized as follows. In Section 1, we introduce a multispecies one-dimensional stationary fluid model that can be viewed as a singular perturbation problem for a multifluid Euler-Poisson type system. The paper is devoted to the study of the formal limit problem called plasma approximation which models the neutral plasma from which the ion beam is extracted. In Section 2, we show that solving this limit model amounts to solve a highly singular Cauchy problem for a nonlinear differential system. We state an

existence result and describe qualitative properties of the solution of this Cauchy problem. In particular, the solution is shown to exist on a maximal interval $[0, x_0)$: x_0 characterizes the limit of the neutral plasma at which the ion beam is extracted. In Section 3, we establish a priori bounds for the solution of the model. Using these estimates, we give in Section 4 the proof of the result of Section 2. In Section 5, we present a numerical method of solution of the plasma approximation and we compare the results obtained for this model with those obtained for a multispecies kinetic model and for a one velocity model.

2. The model.

We consider a simple device (cf. [10]): an *unmagnetized* plasma is generated between two parallel plane absorbing electrodes at the same potential. The device is thus symmetric with respect to the plane $X = 0$, the electrodes being located at $X = \pm a$. Moreover, we can use a one-dimensional plane modeling. We suppose that the plasma consists of electrons with charge - e and of p species of ions, indexed by $\alpha = 1, \cdots, p$, with mass m_α and charge $q_\alpha = Z_\alpha e$. We assume that the electrons behave as an isothermal fluid with temperature T_e. Then by neglecting the inertia of electrons in the electron momentum conservation equation, we obtain that the electron density N_e is related to the electric potential Φ by the Maxwell - Boltzmann relation

$$N_e = C \exp\left(\frac{e\Phi}{kT_e}\right)$$

where C is a constant. Denoting by N_0 the electron density and setting $\Phi = 0$ at the plasma center $X = 0$ we obtain

$$N_e = N_0 \exp\left(\frac{e\Phi}{kT_e}\right). \tag{2.1}$$

On the other hand, we assume that the ions of the species α are formed at rest with an ionization rate $G_\alpha = G_\alpha(N_e)$ which depends only on the electron density. Typically, we have

$$G_\alpha(N_e) = \nu_\alpha N_0 \left(\frac{N_e}{N_0}\right)^{\gamma_\alpha}$$

where ν_α is a collision frequency and $\gamma_\alpha \geq 0$ is a constant which characterizes the ionization process ($\gamma_\alpha = 0, 1, 2$ in practice). In addition, we suppose that the ions are non collisional and that we can neglect the temperature of each species. Assuming stationarity, we obtain that the density N_α and the velocity U_α of the ions of the species α satisfy the equations

$$\frac{d}{dX}\,(N_\alpha U_\alpha) = G_\alpha(N_e), \tag{2.2}$$

$$m_\alpha \frac{d}{dX}\,(N_\alpha U_\alpha^2) = -Z_\alpha e N_\alpha \frac{d\Phi}{dX}. \tag{2.3}$$

Finally, the electric potential Φ satisfies the Poisson equation

$$-\frac{d^2\Phi}{dX^2} = \frac{e}{\varepsilon_0}\,\left(\sum_{\alpha=1}^{p} Z_\alpha N_\alpha - N_e\right). \tag{2.4}$$

We supplement the above equations (2.1)-(2.4) with the following boundary conditions which reflect the symmetry of the device

$$U_\alpha(0) = 0\ ,\ 1 \leq \alpha \leq p, \tag{2.5}$$

$$\Phi(0) = \frac{d\Phi}{dX}(0) = 0. \tag{2.6}$$

The problem (2.1)-(2.6) has indeed to be viewed as a singular perturbation problem. This becomes clear when performing a scaling of the above equations. We introduce characteristic quantities:

- a length $L = \frac{1}{\nu}\sqrt{\frac{kT_e}{m}}$ where ν is a typical ionization frequency and m a typical ion mass,
- an electric potential $\bar{\Phi} = -\frac{kT_e}{e}$,
- an electron density $\bar{N}_e = N_0$,

and for each ion's species α

- a density $\bar{N}_\alpha = \frac{N_0}{Z_\alpha}$
- a velocity $\bar{U}_\alpha = \sqrt{\frac{Z_\alpha kT_e}{m_\alpha}}$,
- an ionization rate $\bar{G}_\alpha = \frac{1}{L}\sqrt{\frac{kT_e}{Z_\alpha m_\alpha}}\,\bar{N}_e = \nu\sqrt{\frac{Z_\alpha m}{m_\alpha}}\,\bar{N}_e$.

If we define the dimensionless quantities $x, \varphi, n_\alpha, n_e, u_\alpha$ by

$$X = Lx, \Phi = \bar{\Phi}\varphi\ ,\ N_e = \bar{N}_e n_e\ ,\ N_\alpha = \bar{N}_\alpha n_\alpha\ ,\ U_\alpha = \bar{U}_\alpha u_\alpha$$

and we set

$$g_\alpha(n_e) = \frac{1}{\bar{G}_\alpha}\, G_\alpha(\bar{N}_e n_e),$$

the equations (2.1)-(2.4) become respectively

$$n_e = \exp(-\varphi), \tag{2.7}$$

$$\frac{d}{dx}\,(n_\alpha u_\alpha) = g_\alpha(n_e)\ ,\ 1 \le \alpha \le p, \tag{2.8}$$

$$\frac{d}{dx}\,(n_\alpha u_\alpha^2) = n_\alpha \frac{d\varphi}{dx}\ ,\ 1 \le \alpha \le p, \tag{2.9}$$

$$\varepsilon^2\,\frac{d^2\varphi}{dx^2} = \sum_{\alpha=1}^{p} n_\alpha - n_e \tag{2.10}$$

where

$$\varepsilon = \frac{\nu}{\omega_p} = \frac{\lambda_D}{L} \quad , \quad \lambda_D = \sqrt{\frac{\varepsilon_0 kT_e}{N_0 e^2}}, \quad \omega_p^2 = \frac{N_0 e^2}{\varepsilon_0 m},$$

while the boundary conditions (2.5),(2.6) give

$$u_\alpha(0) = 0\ ,\ 1 \le \alpha \le p, \tag{2.11}$$

$$\varphi(0) = \frac{d\varphi}{dx}(0) = 0. \tag{2.12}$$

For the physical validity of the model, we refer to [14,10] in which the case of a single ion species is considered. This model is in fact the basis for the numerical simulation of ion extraction (cf [15,16])

In most of the physical situations the Debye length λ_D is far smaller than the ionization characteristic length L so that $\varepsilon > 0$ is a small parameter and the initial value problem (2.7)-(2.12) is indeed a singular perturbation problem. In fact, this problem involves two essential mathematical difficulties(see for example [5,9]):

(i) It presents a singularity at the origin and the authors are not aware of any existence result of a solution at least in the multispecies case $p \geq 2$ although extensive numerical computations have shown that for any $\varepsilon > 0$ such a solution exists and is uniquely defined on the whole half line $x \geq 0$. Note however that an existence result has been proved in the case $p = 1$ (cf. [1], [12]);

(ii) As we shall see it in Section 2, the formal limit problem corresponding to $\varepsilon = 0$ has a solution which is defined only on a finite interval $[0, x_0)$. As a consequence, the asymptotic analysis of the problem (2.7)-(2.12) appears to be far from being standard and seems to need new ideas and techniques.

This paper is devoted to the study of this formal limit problem or *plasma approximation* in the physicists'terminology. It may be considered as a first step towards the mathematical analysis of the singular perturbation problem (2.7)-(2.12). This formal limit consists in setting $\varepsilon = 0$ in (2.10); we obtain the condition

$$n_e = \sum_{\alpha=1}^{p} n_\alpha \tag{2.13}$$

which expresses that the plasma is locally electrically neutral. For deriving the limit model, we eliminate the electric potential: using (2.7), we obtain

$$\frac{d\varphi}{dx} = -\frac{1}{n_e}\,\frac{dn_e}{dx},$$

and

$$\varphi(0) = 0 \iff n_e(0) = 1.$$

Hence the plasma approximation amounts to find $\{(n_\alpha, u_\alpha); 1 \leq \alpha \leq p\}$, solutions of the differential equations

$$\frac{d}{dx}\,(n_\alpha u_\alpha) = g_\alpha(n_e)\ ,\ 1 \leq \alpha \leq p, \tag{2.14}$$

$$\frac{d}{dx}\,(n_\alpha u_\alpha^2) + \frac{n_\alpha}{n_e}\,\frac{dn_e}{dx} = 0\ ,\ 1 \leq \alpha \leq p, \tag{2.15}$$

with the initial conditions

$$n_e(0) = 1, \tag{2.16}$$

$$u_\alpha(0) = 0\ ,\ 1 \leq \alpha \leq p, \tag{2.17}$$

and the neutrality condition (2.13).

3. The plasma approximation.

Before establishing the existence of a solution of the limit model (2.13)-(2.17) and studying its properties, we need to put this problem in a more convenient form. We set

$$j_\alpha = n_\alpha u_\alpha, \quad k_\alpha = n_\alpha u_\alpha^2 \tag{3.1}$$

where j_α and k_α represent respectively the scaled current and kinetic energy of the α species, so that the equations (2.14), (2.15) are respectively written

$$\frac{dj_\alpha}{dx} = g_\alpha(n_e), \tag{3.2}$$

$$\frac{dk_\alpha}{dx} + \frac{n_\alpha}{n_e}\frac{dn_e}{dx} = 0. \tag{3.3}$$

By replacing k_α by $\dfrac{j_\alpha^2}{n_\alpha}$ and using (2.13) and (3.2), Eq.(3.3) becomes

$$-u_\alpha^2 \frac{dn_\alpha}{dx} + \frac{n_\alpha}{n_e} \sum_{\beta=1}^{p} \frac{dn_\beta}{dx} = -2u_\alpha g_\alpha(n_e).$$

Next, we introduce the vectors of dimension $2p$

$$\boldsymbol{U} = \left(\begin{array}{c} n_1 \\ \vdots \\ n_p \\ \hline j_1 \\ \vdots \\ j_p \end{array}\right), \qquad \boldsymbol{G}(\boldsymbol{U}) = \left(\begin{array}{c} -2u_1\, g_1(n_e) \\ \vdots \\ -2u_p\, g_p(n_e) \\ \hline g_1(n_e) \\ \vdots \\ g_p(n_e) \end{array}\right)$$

and the $2p \times 2p$ matrix

$$\boldsymbol{A}(\boldsymbol{U}) = \left(\begin{array}{c|c} \boldsymbol{B}(\boldsymbol{U}) & \mathbf{0} \\ \hline \mathbf{0} & \boldsymbol{I} \end{array}\right)$$

where $\boldsymbol{B}(\boldsymbol{U})$ is the $p \times p$ matrix

$$\boldsymbol{B}(\boldsymbol{U}) = \begin{pmatrix} \frac{n_1}{n_e} - u_1^2 & \frac{n_1}{n_e} & \dots & \frac{n_1}{n_e} \\ \frac{n_2}{n_e} & \frac{n_2}{n_e} - u_2^2 & \dots & \frac{n_2}{n_e} \\ \vdots & \vdots & & \vdots \\ \frac{n_p}{n_e} & \frac{n_p}{n_e} & \dots & \frac{n_p}{n_e} - u_p^2 \end{pmatrix}$$

so that differential system (2.14), (2.15) may be equivalently written

$$\boldsymbol{A}(\boldsymbol{U})\, \frac{d\boldsymbol{U}}{dx} = \boldsymbol{G}(\boldsymbol{U}) \tag{3.4}$$

with again the neutrality condition (2.13). The initial conditions (2.16) and (2.17) give the following $p+1$ constraints

$$\begin{cases} n_e(0) = 1 \\ \\ j_\alpha(0) = 0\ ,\ 1 \leq \alpha \leq p \end{cases} \tag{3.5}$$

and do not specify the whole initial condition $\boldsymbol{U}(0)$ in the multispecies case $p \geq 2$. On the other hand, the matrix $\boldsymbol{A}(\boldsymbol{U})$ may be singular. The following result shows in particular that this is indeed the case at $x = 0$.

Lemma 1. We have

$$\det(\boldsymbol{A}(\boldsymbol{U})) = \frac{(-1)^{p-1}}{n_e} \left(\prod_{\alpha=1}^{p} u_\alpha^2 \right) \left(\sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2} - n_e \right). \tag{3.6}$$

Proof. We have obviously

$$\det(\boldsymbol{A}(\boldsymbol{U})) = \det(\boldsymbol{B}(\boldsymbol{U})).$$

Let us prove by induction that $\det(\boldsymbol{B}(\boldsymbol{U}))$ is given by the right hand side of (3.6). The result is clearly true for $p = 1$. Assume that it holds for $p - 1$. By developing $\det(\boldsymbol{B}(\boldsymbol{U}))$ along its first line, we find

$$\det(\boldsymbol{B}(\boldsymbol{U})) = \left(\frac{n_1}{n_e} - u_1^2 \right) \begin{vmatrix} \frac{n_2}{n_e} - u_2^2 & \dots & \frac{n_2}{n_e} \\ \vdots & & \vdots \\ \frac{n_p}{n_e} & \dots & \frac{n_p}{n_e} - u_p^2 \end{vmatrix} -$$

$$- \frac{n_1}{n_e} \begin{vmatrix} \frac{n_2}{n_e} & \frac{n_2}{n_e} & \dots & \frac{n_2}{n_e} \\ \frac{n_3}{n_e} & \frac{n_3}{n_e} - u_3^2 & \dots & \frac{n_3}{n_e} \\ \vdots & \vdots & & \vdots \\ \frac{n_p}{n_e} & \frac{n_p}{n_e} & \dots & \frac{n_p}{n_e} - u_p^2 \end{vmatrix}$$

$$+ \dots + (-1)^{p-1} \frac{n_1}{n_e} \begin{vmatrix} \frac{n_2}{n_e} & \frac{n_2}{n_e} - u_2^2 & \dots & \frac{n_2}{n_e} \\ \vdots & \vdots & & \vdots \\ \frac{n_{p-1}}{n_e} & \frac{n_{p-1}}{n_e} & \dots & \frac{n_{p-1}}{n_e} - u_{p-1}^2 \\ \frac{n_p}{n_e} & \frac{n_p}{n_e} & \dots & \frac{n_p}{n_e} \end{vmatrix}.$$

Now, we use the induction hypothesis for evaluating the first determinant and we compute the remaining determinants by substracting the first column for the others. We obtain

$$\begin{aligned} \det(\boldsymbol{B}(\boldsymbol{U})) &= (-1)^{p-2} \left(\frac{n_1}{n_e} - u_1^2 \right) \frac{1}{n_e} \left(\prod_{\alpha=2}^{p} u_\alpha^2 \right) \left(\sum_{\alpha=2}^{p} \frac{n_\alpha}{u_\alpha^2} - n_e \right) + \\ &+ (-1)^{p-1} \left\{ \frac{n_1 n_2}{n_e^2} u_3^2 \cdots u_p^2 + \cdots + \frac{n_1 n_p}{n_e^2} u_2^2 \cdots u_{p-1}^2 \right\} = \\ &= \frac{(-1)^{p-1}}{n_e^2} \left(\prod_{\alpha=1}^{p} u_\alpha^2 \right) \left\{ \left(-\frac{n_1}{u_1^2} + n_e \right) \left(\sum_{\alpha=2}^{p} \frac{n_\alpha}{u_\alpha^2} - n_e \right) + \frac{n_1}{u_1^2} \left(\sum_{\alpha=2}^{p} \frac{n_\alpha}{u_\alpha^2} \right) \right\} \end{aligned}$$

and (3.6) follows.□

As a consequence of (3.6), we obtain that the matrix $\boldsymbol{A}(\boldsymbol{U})$ is singular if and only if one of the two following situations holds:
(i) $p \geq 2$ and $u_\alpha = 0$ for at least two indices α;
(ii) we have

$$\sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2} = n_e. \tag{3.7}$$

In particular, in the multispecies case $p \geq 2$, the initial conditions (2.17) imply that the matrix $\boldsymbol{A}(\boldsymbol{U}(0))$ is singular whatever the $n_\alpha(0)'$s may be. In fact, since $\boldsymbol{B}(\boldsymbol{U}(0))$ is a matrix of rank 1, we find that 0 is an eigenvalue of $\boldsymbol{A}(\boldsymbol{U}(0))$ of multiplicity $p-1$ with a corresponding eigenspace of dimension 1.

In all the sequel, we will call (physically admissible) solution of the initial value problem (2.13)-(2.17) or (3.4), (3.5) any C^1 function $\boldsymbol{U}$ from an interval $I = [0, x_1]$ into $\mathbf{R}^{2p}$ which satisfies the equations (3.4), (3.5) and the physical constraints

$$n_\alpha(x) > 0 \quad , \ \alpha \leq 1 \leq p. \tag{3.8}$$

We will also assume the following hypothesis.

$$\left\{ \begin{array}{l} \text{each (nondimensional) ionizate rate } g_\alpha : \mathbf{R}_+ - \mathbf{R}_+ \text{ is an increasing} \\ C^1 \text{ function which satisfies} \qquad g_\alpha(n_e) > 0 \quad \forall n_e > 0 \quad (^*). \end{array} \right. \tag{3.9}$$

We set

$$g(n_e) = \sum_{\alpha=1}^{p} g_\alpha(n_e). \tag{3.10}$$

Then, we can state the main result of this paper:

Theorem 2. Assume the hypothesis (3.9). The plasma approximation model (2.13)-(2.17) has a (physically admissible) solution defined in a maximal interval $[0, x_0)$ with $x_0 < +\infty$, which possesses the following properties:
(i) $n_e \in C^2([0, x_0))$ is a strictly decreasing function;
(ii) we have at $x = 0$

$$n_\alpha(0) = \frac{g_\alpha(1)}{g(1)} \quad , \ \frac{dn_\alpha}{dx}(0) = 0, \ 1 \leq \alpha \leq p, \tag{3.11}$$

$$\frac{dn_e}{dx}(0) = 0 \quad , \quad \frac{d^2 n_e}{dx^2}(0) = -2g(1)^2; \tag{3.12}$$

(iii) $\boldsymbol{U}(x_0) = \lim_{x \to x_0} \boldsymbol{U}(x)$ exists;
(iv) the matrix $\boldsymbol{A}(\boldsymbol{U}(x_0))$ is singular and we have at $x = x_0$

$$\left\{ \begin{array}{l} \displaystyle\sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2} = n_e \\ \displaystyle\sum_{\alpha=1}^{p} n_\alpha u_\alpha^2 = 1 - n_e \end{array} \right. \tag{3.13}$$

*This hypothesis, which is physically realistic, can be slightly weakened mathematically.

and

$$n_e(x_0) \leq \frac{1}{2}; \tag{3.14}$$

(v) we have

$$\lim_{x \to x_0} \frac{dn_\alpha}{dx}(x) = -\infty \quad , \quad \lim_{x \to x_0} \frac{du_\alpha}{dx}(x) = +\infty \quad , \; 1 \leq \alpha \leq p. \tag{3.15}$$

We conjecture that this solution is unique (cf. Remark 3). Let us illustrate the above result by considering two simple but useful examples.

Example 1. *The case of a single ion species*. If $p = 1$, the system (2.13)-(2.17) reduces to

$$\begin{cases} \frac{d}{dx}(nu) = g(n), \\ \frac{d}{dx}(nu^2 + n) = 0, \\ n(0) = 1, \;\; u(0) = 0, \end{cases} \tag{3.16}$$

where n denotes the density of both electrons and ions and u is the ion velocity. This system (3.16) can be easily solved. Indeed, using the second equation (3.16) and the initial conditions, we obtain

$$nu^2 + n = 1$$

and therefore

$$u^2 = \frac{1-n}{n}$$

which yields $n \leq 1$. On the other hand, the first equation (3.16) together with the initial conditions yield

$$(nu)(x) = \int_0^x g(n(y))dy \geq 0.$$

Hence u is ≥ 0 and we have

$$u = \sqrt{\frac{1-n}{n}}.$$

By replacing u by its value in the first equation (3.16), we find

$$\frac{d}{dx}\sqrt{n(1-n)} = g(n)$$

so that (3.16) amounts to solve the equation

$$\int_1^{n(x)} \frac{(1-2m)dm}{2g(m)\sqrt{m(1-m)}} = x. \tag{3.17}$$

Next, we notice that the function

$$f(n) = \int_1^n \frac{(1-2m)dm}{2g(m)\sqrt{m(1-m)}}$$

has in the interval [0, 1] a unique maximum $x_0 = f\left(\frac{1}{2}\right)$ at $n = \frac{1}{2}$ and is increasing in $\left[0, \frac{1}{2}\right]$ then decreasing in $\left[\frac{1}{2}, 1\right]$. Therefore the equation (3.17) has a solution only for $x \leq x_0$. Since $n(0) = 1$, we obtain that the function $x \to n(x)$ decreases from 1 to $\frac{1}{2}$ as x increases from 0 to x_0. Moreover, we have clearly

$$\lim_{x \to x_0} \frac{dn}{dx}(x) = -\infty.$$

We thus get conclusions of the Theorem 2 with $n(x_0) = \frac{1}{2}$. Moreover we have

$$u(x_0) = 1\ , \quad j(x_0) = (nu)(x_0) = \frac{1}{2}. \quad \square \tag{3.18}$$

Note that for a constant ionization rate $g = g_0$, the exact solution is given by

$$n(x) = \frac{1 + \sqrt{1 - 4g_0^2 x^2}}{2}.$$

Example 2. *A particular case of multispecies ionization.* We next consider the case -which will be useful in the proof of the Theorem- where the ionization rates are proportional, i.e.,

$$g_\alpha(n_e) = a_\alpha g(n_e)\ ,\ 1 \leq \alpha \leq p, \tag{3.19}$$

for some constants $a_\alpha > 0,\ 1 \leq \alpha \leq p$, with $\displaystyle\sum_{\alpha=1}^{p} a_\alpha = 1$. Then, denoting by (n, u) the solution of (3.16), it is an obvious matter to check that

$$n_\alpha = a_\alpha n\ , \quad u_\alpha = u\ , \quad 1 \leq \alpha \leq p, \tag{3.20}$$

is solution of the system (2.13)-(2.17). Let us prove that this is the only solution. When (3.19) holds, (2.14) is written:

$$\frac{d}{dx}(n_\alpha u_\alpha) = a_\alpha g(n_e),$$

which yields

$$n_\alpha u_\alpha = a_\alpha j\ , \quad j(x) = \int_0^x g(n_e(y))dy.$$

Replacing in (2.15) n_α by $\dfrac{a_\alpha j}{u_\alpha}$ gives

$$\frac{d}{dx}(ju_\alpha) + \frac{1}{u_\alpha}\,\frac{1}{n_e}\,\frac{dn_e}{dx} = 0,$$

and therefore

$$\frac{d}{dx}(ju_\alpha)^2 = -\frac{2j}{n_e}\,\frac{dn_e}{dx}.$$

Since $(ju_\alpha)(0) = 0$, we obtain that u_α is independent of α. Hence we have for some function u

$$u_\alpha = u\ ,\ n_\alpha = a_\alpha n\ ,\ n = \frac{j}{u}.$$

Moreover the pair (n, u) is easily seen to be the solution of (3.16) which proves our assertion.$\square$

Let us conclude this section by giving a physical picture of the results of Theorem 1. Going back to the device considered in Section 1, we obtain that the plasma approximation is valid in a slab of maximal width $2Lx_0$. Then two cases are to be considered.
(i) If $a \le X_0 = Lx_0$, the plasma is quasineutral in the whole interelectrode domain at the exception of thin layers located at each electrode $X = \pm\, a$. We refer to [4], [3], [13] for an analysis of the corresponding boundary layer problem in the absence of the ionization source terms.

(ii) If $a > X_0$, the plasma is quasi neutral in the slab $|X| \le X_0$. At X_0, the electric neutrality breaks down: ions are extracted from the plasma while the electrons remains confined in the neutral plasma zone. We refer to [11,10] for a physical description of ion extraction from a plasma an to [2,13] for a mathematical discussion of a related but simpler situation.

In the case of one single ion species, the velocity $U(X_0)$ and the current density $J(X_0) = Ze(NU)(X_0)$ of the ions leaving the neutral plasma at X_0 are independent of the ionization rate:

$$U(X_0) = \sqrt{\frac{kT_e}{m}} \quad , \quad J(X_0) = \frac{N_0 Ze}{2}\sqrt{\frac{kT_e}{m}}.$$

These expressions have been indeed obtained by the physicists (see [10]) and are known respectively as the ion acoustic velocity and the Bohm current.

The picture is not so simple in the multispecies case: the velocity $U_\alpha(X_0)$ and the current density $J_\alpha(X_0) = Z_\alpha e(N_\alpha U_\alpha)(X_0)$ of the ions of the species α leaving the plasma at X_0 depend now on the ionization rates. However as we will see it in Section 5, extensive numerical simulations indicate that we have in any case

$$u_\alpha(x_0) \simeq 1,$$

and therefore by (3.13)

$$j(x_0) \simeq \frac{1}{2}.$$

This yields

$$U_\alpha(X_0) \simeq \sqrt{\frac{Z_\alpha kT_e}{m_\alpha}} \quad , \quad \sum_{\alpha=1}^{p} \frac{J_\alpha(X_0)}{\sqrt{\frac{Z_\alpha kT_e}{m_\alpha}}} \simeq \frac{eN_0}{2},$$

where $J_\alpha = Z_\alpha e N_\alpha U_\alpha$. However, we have no further a priori information on the partial current densities J_α which are of practical importance for the physicists.

4. A priori estimates.

As a preliminary step for proving Theorem 2, we first derive a priori bounds for any solution of the problem (2.13)-(2.17) (or (3.4),(3.5)). Using (3.2) and (2.17), we have

$$j_\alpha(x) = \int_0^x g_\alpha(n_e(y))dy. \tag{4.1}$$

Then it follows from the hypothesis (3.9) that

$$j_\alpha(x) > 0 \quad \text{for all } x > 0\ , \quad 1 \le \alpha \le p. \tag{4.2}$$

Since

$$k_\alpha = \frac{j_\alpha^2}{n_\alpha}$$

we find by (3.8)

$$k_\alpha(x) > 0 \quad \text{for all } x > 0\ , \quad 1 \le \alpha \le p. \tag{4.3}$$

The following result will play an essential role in all the sequel.

Lemma 3. Any solution of the system (2.13)-(2.17) satisfies the inequality

$$\frac{dn_e}{dx}(x) < 0 \tag{4.4}$$

at all point $x > 0$ where this solution exists.

Proof. By multiplying (3.3) by k_α, we obtain

$$\frac{1}{2}\frac{d}{dx}k_\alpha^2 = -\frac{n_\alpha k_\alpha}{n_e}\frac{dn_e}{dx} = -\frac{j_\alpha^2}{n_e}\frac{dn_e}{dx}$$

and since $k_\alpha(0) = 0$

$$k_\alpha^2(x) = -2\int_0^x \left(\frac{j_\alpha^2}{n_e}\frac{dn_e}{dx}\right)(y)dy. \tag{4.5}$$

Then it follows from (4.3) that

$$\int_0^x \frac{j_\alpha^2}{n_e}\frac{dn_e}{dx}(y)dy < 0 \quad \text{for all } x < 0.$$

Together with (4.2), this implies that (4.4) holds for $x > 0$ small enough. In order to prove (4.4) for any $x > 0$, it is enough to check that $\frac{dn_e}{dx}$ cannot vanish. Indeed, using (2.13), we can write

$$\frac{dn_e}{dx} = \frac{d}{dx}\left(\sum_{\alpha=1}^p n_\alpha\right) = \frac{d}{dx}\left(\sum_{\alpha=1}^p \frac{j_\alpha^2}{k_\alpha}\right) = \sum_{\alpha=1}^p \left(2\,\frac{j_\alpha}{k_\alpha}\frac{dj_\alpha}{dx} - \frac{j_\alpha^2}{k_\alpha^2}\frac{dk_\alpha}{dx}\right)$$

so that by (3.2),(3.3)

$$\frac{dn_e}{dx} = \sum_{\alpha=1}^{p} \left(2\,\frac{j_\alpha}{k_\alpha}\, g_\alpha(n_e) + \frac{j_\alpha^2}{k_\alpha^2}\,\frac{n_\alpha}{n_e}\,\frac{dn_e}{dx} \right).$$

Hence we obtain

$$\left(1 - \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha^2}\,\frac{n_\alpha}{n_e}\right)\frac{dn_e}{dx} = 2\sum_{\alpha=1}^{p} \frac{j_\alpha}{k_\alpha} g_\alpha(n_e). \tag{4.6}$$

Since by (3.9), (4.2), (4.3), we have

$$\frac{j_\alpha}{k_\alpha} g_\alpha(n_e) > 0 \quad \text{for all } x > 0,$$

we find

$$\left(1 - \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha^2}\,\frac{n_\alpha}{n_e}\right)\frac{dn_e}{dx} > 0 \quad \text{for all } x > 0 \tag{4.7}$$

which proves our assertion. □

Remark 3. The inequality (4.6) can be also written

$$\left(n_e - \sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2}\right)\frac{dn_e}{dx} > 0 \quad \text{for all } x > 0. \tag{4.8}$$

Then it follows from (3.6) that the matrix $\boldsymbol{A}(\boldsymbol{U}(x))$ is never singular at a point $x > 0$ where the solution $\boldsymbol{U}$ of (3.4), (3.5) is defined. Hence, at such a point $x > 0$, (3.4) becomes

$$\frac{d\boldsymbol{U}}{dx} = \boldsymbol{F}(\boldsymbol{U})\,, \quad \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{A}(\boldsymbol{U})^{-1}\boldsymbol{G}(\boldsymbol{U})$$

and $\boldsymbol{F}$ is a C^1 function. Therefore the uniqueness of the solution of (2.13)-(2.17) holds as soon as one can prove the uniqueness of the solution in a neighborhood of $x = 0$. □

Lemma 4. We have:

$$n_a \le n_e \le 1\,, \quad 1 \le \alpha \le p \tag{4.9}$$

and

$$j = \sum_{\alpha=1}^{p} j_\alpha \le \frac{1}{2} \tag{4.10}$$

Proof. The bounds (4.9) follow at once from Lemma 3 and the initial condition (2.16). Let us check (4.10). By summing (3.3) with respect to α and using (2.13), we obtain the conservation law

$$\frac{d}{dx}\left(\sum_{\alpha=1}^{p} k_\alpha + n_e\right) = 0$$

and by integration from 0 to x

$$\sum_{\alpha=1}^{p} k_\alpha = 1 - n_e. \tag{4.11}$$

On the other hand, the neutrality condition (2.13) can be equivalently written

$$\sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha} = n_e. \tag{4.12}$$

Then summing (4.11) and (4.12) gives

$$\sum_{\alpha=1}^{p} j_\alpha \left(\frac{j_\alpha}{k_\alpha} + \frac{k_\alpha}{j_\alpha} \right) = 1.$$

Since by (4.2) and (4.3)

$$\frac{j_\alpha}{k_\alpha} > 0 \quad \text{for } x > 0$$

and

$$a + \frac{1}{a} \geq 2 \quad \text{for all } a > 0,$$

this implies the bound (4.10). □

Next, we want to make more precise the behavior of n_e in a neighborhood of $x = 0$. In all the sequel the following function

$$h(x) = -2 \int_0^x y^2 \left(\frac{1}{n_e} \frac{dn_e}{dx} \right)(y) dy \tag{4.13}$$

will be frequently used. We can then state

Lemma 5. For all $x \geq 0$ such that $n_e(x) \geq \frac{1}{2}$, we have

$$1 - 4g(1)^2 x^2 \leq n_e(x) \leq 1 - g\Big(\frac{1}{2}\Big) \left\{ \max_{1 \leq \alpha \leq p} \frac{g_\alpha(\frac{1}{2})^2}{g_\alpha(1)} \right\} x^2 \tag{4.14}$$

and

$$g_\alpha\Big(\frac{1}{2}\Big) \left\{ \max_{1 \leq \beta \leq p} \frac{g_\beta(\frac{1}{2})^2}{g_\beta(1)} \right\} x^2 \leq k_\alpha(x) \leq 4 g_\alpha(1) g(1) x^2. \tag{4.15}$$

Proof. Since g_α is an increasing function, we have for $n_e \in \left[\frac{1}{2}, 1\right]$

$$g_\alpha\big(\frac{1}{2}\big) \leq g_\alpha(n_e) \leq g_\alpha(1)$$

and by (4.1)

$$g_\alpha\left(\tfrac{1}{2}\right)x \leq j_\alpha(x) \leq g_\alpha(1)x. \tag{4.16}$$

Then, using the bounds (4.4) and (4.16) together with (4.13), (4.5) yields

$$g_\alpha\left(\tfrac{1}{2}\right)\sqrt{h} \leq k_\alpha \leq g_\alpha(1)\sqrt{h} \tag{4.17}$$

so that we obtain by (4.11)

$$1 - g(1)\sqrt{h} \leq n_e \leq 1 - g\left(\tfrac{1}{2}\right)\sqrt{h}. \tag{4.18}$$

It remains to derive upper and lower bounds for h. We begin with the upper bound. Assuming $n_e(x) \geq \frac{1}{2}$, we have

$$h(x) \leq -4\int_0^x y^2 \frac{dn_e}{dx}(y)dy$$

and therefore,

$$h(x) \leq 4(1 - n_e(x))x^2. \tag{4.19}$$

Together with the first inequality (4.18), this implies the first inequality (4.14) and therefore

$$h(x) \leq \ 16\, g(1)^2 x^4. \tag{4.20}$$

For obtaining a lover bound for h, we start from

$$n_e = \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha} \leq 1$$

which yields

$$k_\alpha \geq j_\alpha^2\ ,\ 1 \leq \alpha \leq p.$$

Using again (4.5) and the estimates (4.16), (4.17), we obtain

$$g_\alpha(1)^2 h \geq k_\alpha^2 \geq j_\alpha^4 \geq g_\alpha\left(\tfrac{1}{2}\right)^4 x^4\ ,\ 1 \leq \alpha \leq p$$

which yields

$$h(x) \geq \ \left\{\max_{1\leq\alpha\leq p} \frac{g_\alpha(\frac{1}{2})^4}{g_\alpha(1)^2}\right\} x^4. \tag{4.21}$$

Now the second bound (4.14) and the bounds (4.15) follow from (4.17),(4.18),(4.20) and (4.21).□

Let us next estimate the first and second derivatives of n_e in a neighborhood of $x = 0$.
Lemma 6. There exists a constant $C > 0$ depending only on $g_\alpha(\frac{1}{2})$, $g_\alpha(1)$, $1 \leq \alpha \leq p$, such that we have for $x \geq 0$ small enough

$$-\frac{dn_e}{dx}(x) \leq Cx \tag{4.22}$$

Proof. Observe that (4.6) may be equivalently written

$$-\frac{dn_e}{dx} = \frac{A}{B} \tag{4.23}$$

with

$$A = 2n_e \sum_{\alpha=1}^{p} \frac{j_\alpha}{k_\alpha}\, g_\alpha(n_e), \tag{4.24}$$

$$B = \sum_{\alpha=1}^{p} \frac{j_\alpha^4}{k_\alpha} - n_e. \tag{4.25}$$

Denoting by $C_i > 0$, $i = 1, 2, \cdots$, various constants depending only on $g_\alpha(\frac{1}{2})$, $g_\alpha(1)$, $1 \le \alpha \le p$, and using the bounds (4.9),(4.15) and (4.16) we obtain on one hand

$$A \le \frac{C_1}{x}. \tag{4.26}$$

On the other hand, we find

$$\sum_{\alpha=1}^{p} \frac{j_\alpha^4}{k_\alpha^3} \ge \frac{C_2}{x^2},$$

and for x small enough

$$B \ge \frac{C_3}{x^2}. \tag{4.27}$$

The estimate (4.22) follows. □

Note that all the previous bounds hold if the g'_αs are C^0 functions. In order to obtain an estimate for $\dfrac{d^2n_e}{dx^2}$, we need to assume more regularity on the functions g_α: we thus suppose as in (3.9) that each g_α is a C^1 function but it would be enough to assume $g_\alpha \in W^{1,\infty}(0,1)$.

Lemma 7. We have for $x \ge 0$ small enough

$$\left| \frac{d^2n_e}{dx^2}\,(x) \right| \le C \tag{4.28}$$

where $C > 0$ is a constant which depends only on $g_\alpha\big(\frac{1}{2}\big)$, $g_\alpha(1)$ and $||g'_\alpha||_{L^\infty(\frac{1}{2},1)}$, $1 \le \alpha \le p$.

Proof. We proceed as in the previous lemma. Differentiating (4.23) gives

$$-\frac{d^2n_e}{dx^2} = \frac{1}{B}\,\frac{dA}{dx} - \frac{A}{B^2}\,\frac{dB}{dx}. \tag{4.29}$$

Hence we need only to estimate $\left|\dfrac{dA}{dx}\right|$ and $\left|\dfrac{dB}{dx}\right|$. We have

$$\begin{aligned}\frac{dA}{dx} &= 2\frac{dn_e}{dx}\sum_{\alpha=1}^{p}\frac{j_\alpha g_\alpha}{k_\alpha} + 2n_e\sum_{\alpha=1}^{p}\left\{\frac{g_\alpha}{k_\alpha}\,\frac{dj_\alpha}{dx} + \frac{j_\alpha}{k_\alpha}\,g'_\alpha\,\frac{dn_e}{dx} - \frac{j_\alpha g_\alpha}{k_\alpha^2}\,\frac{dk_\alpha}{dx}\right\}\\ &= 2\frac{dn_e}{dx}\sum_{\alpha=1}^{p}\frac{j_\alpha g_\alpha}{k_\alpha} + 2n_e\sum_{\alpha=1}^{p}\left\{\frac{g_\alpha^2}{k_\alpha} + \frac{j_\alpha}{k_\alpha}g'_\alpha\,\frac{dn_e}{dx} + \frac{j_\alpha^3 g_\alpha}{k_\alpha^3}\,\frac{1}{n_e}\,\frac{dn_e}{dx}\right\}\end{aligned}$$

which yields

$$\left|\frac{dA}{dx}\right| \le \frac{C_1}{x^2}. \tag{4.30}$$

Similarly, we can write

$$\begin{aligned}\frac{dB}{dx} &= \sum_{\alpha=1}^{p}\left\{\frac{4j_\alpha^3}{k_\alpha^3}\,\frac{dj_\alpha}{dx} - \frac{3j_\alpha^4}{k_\alpha^4}\,\frac{dk_\alpha}{dx}\right\} - \frac{dn_e}{dx} =\\ &= \sum_{\alpha=1}^{p}\left\{\frac{4j_\alpha^3 g_\alpha}{k_\alpha^3} + \frac{3j_\alpha^6}{k_\alpha^5}\,\frac{1}{n_e}\,\frac{dn_e}{dx}\right\} - \frac{dn_e}{dx}\end{aligned}$$

which yields

$$\left|\frac{dB}{dx}\right| \le \frac{C_2}{x^3}. \tag{4.31}$$

The desired bound (4.28) follows from (4.29) and the inequalities (4.26), (4.27) and (4.30), (4.31).□

5. Proof of Theorem 2.

We begin by characterizing n_e as the solution of a nonlinear integro-differential equation. Using (4.5), we have

$$k_\alpha = \sqrt{-2\int_0^x \left(\frac{j_\alpha^2}{n_e}\,\frac{dn_e}{dx}\right)(y)dy}. \tag{5.1}$$

Hence, writing the neutrality condition (2.13) as

$$n_e = \sum_{\alpha=1}^{p}\frac{j_\alpha^2}{k_\alpha},$$

we obtain that n_e is solution of the equation

$$n_e = \sum_{\alpha=1}^{p}\frac{j_\alpha^2}{\sqrt{-2\int_0^x(\frac{j_\alpha^2}{n_e}\,\frac{dn_e}{dx})(y)dy}} \tag{5.2}$$

where j_α is given in terms of n_e by (4.1).

Now, the proof of Theorem 1 consists of two main steps. In the first step, we show that the conclusions of the theorem hold as soon as one knows a (physically admissible) solution of the problem defined in a neighborhood of $x = 0$. The second step is essentially devoted to the existence of such a local solution.

Let us thus assume that there exists a solution of (2.13)-(2.17) in a neighborhood of $x = 0$. Denote by $[0, x_0)$ the maximal interval of existence of the solution and let us check that we have $x_0 < +\infty$. We proceed by contradiction. Assume in the contrary $x_0 = +\infty$. We observe that, by (3.2) and (3.9), each function j_α is strictly increasing. Moreover using in addition (4.4), we have

$$\frac{d^2 j_\alpha}{dx} = g'_\alpha(n_e)\frac{dn_e}{dx} \leq 0$$

so that j_α is also concave. Then, (4.10) yields

$$\lim_{x\to\infty} j(x) \leq \frac{1}{2}.$$

Let us next show that this is indeed impossible. We first consider the case where $g(0) > 0$. We obtain

$$j(x) = \int_0^x g(n_e(y))dy \geq g(0)x$$

so that $j(x) \geq \frac{1}{2}$ for x large enough which is excluded. We pass to the case where $g(0) = 0$. The function j being strictly increasing and concave, we have necessarily

$$\lim_{x\to\infty} \frac{dj}{dx}(x) = 0,$$

and therefore

$$\lim_{x\to\infty} g(n_e(x)) = 0.$$

Using the hypothesis (3.9), we obtain

$$\lim_{x\to\infty} n_e(x) = 0.$$

On the other hand, since j_α is increasing and bounded above, we have

$$\lim_{x\to\infty} j_\alpha(x) = j_\alpha(\infty) > 0.$$

Similarly, each function k_α is a strictly increasing function since by (3.3),(3.8) and (4.4)

$$\frac{dk_\alpha}{dx}(x) > 0 \quad \text{for all } x > 0.$$

Moreover, it follows from (4.11) that k_α is bounded above so that

$$\lim_{x\to\infty} k_\alpha(x) = k_\alpha(\infty) > 0.$$

Hence we obtain

$$\lim_{x\to\infty} n_e(x) = \sum_{\alpha=1}^{p} \frac{j_\alpha^2(\infty)}{k_\alpha(\infty)} > 0$$

which leads again to a contradiction. Thus, we have necessarily $x_0 < +\infty$.

Let us next check that

$$\boldsymbol{U}(x_0) = \lim_{x\to x_0} \boldsymbol{U}(x)$$

exists. Using again the fact that j_α and k_α are increasing functions which are bounded above, we have

$$\lim_{x\to x_0} j_\alpha(x) = j_\alpha(x_0) > 0\ , \quad \lim_{x\to x_0} k_\alpha(x) = k_\alpha(x_0) > 0$$

and therefore

$$\lim_{x\to x_0} n_\alpha(x) = n_\alpha(x_0) = \frac{j_\alpha^2(x_0)}{k_\alpha(x_0)} > 0$$

which proves our assertion.

The matrix $\boldsymbol{A}(\boldsymbol{U}(x_0))$ is necessarily singular. Otherwise, from the initial condition $\boldsymbol{U}(x_0)$ at x_0, one could extend the solution of (3.4) beyond x_0, which is excluded since $[0, x_0)$ is the maximal interval of existence of the solution. Then it follows from (3.6) that

$$\sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2}(x_0) = n_e(x_0). \tag{5.3}$$

On the other hand, we find by passing to the limit in (4.11)

$$\sum_{\alpha=1}^{p} k_\alpha(x_0) = \sum_{\alpha=1}^{p} (n_\alpha u_\alpha^2)(x_0) = 1 - n_e(x_0).$$

This proves (3.13). As a consequence we have at x_0

$$\sum_{\alpha=1}^{p} n_\alpha(u_\alpha^2 + \frac{1}{u_\alpha^2}) = 1$$

which yields the inequality (3.14).

Let us then prove (3.15). By passing to the limit in (4.8) and using (4.4) together with (5.3), we first obtain

$$\lim_{x\to x_0} \frac{dn_e}{dx}(x) = -\infty$$

and by (3.3)

$$\lim_{x\to x_0} \frac{dk_\alpha}{dx}(x) = -\frac{n_\alpha}{n_e}(x_0) \lim_{x\to x_0} \frac{dn_e}{dx}(x) = +\infty.$$

Since by (3.2)

$$\frac{dk_\alpha}{dx} = \frac{d}{dx}(u_\alpha j_\alpha) = u_\alpha \frac{dj_\alpha}{dx} + j_\alpha \frac{du_\alpha}{dx} = u_\alpha g_\alpha(n_e) + j_\alpha \frac{du_\alpha}{dx}$$

we get

$$\lim_{x\to x_0} \frac{du_\alpha}{dx}(x) = \frac{1}{j_\alpha(x_0)} \left\{ \lim_{x\to x_0} \frac{dk_\alpha}{dx} - u_\alpha(x_0) g_\alpha(n_e(x_0)) \right\} = +\infty.$$

On the other hand, since

$$\frac{dj_\alpha}{dx} = u_\alpha \frac{dn_\alpha}{dx} + n_\alpha \frac{du_\alpha}{dx} = g_\alpha(n_e),$$

we have

$$\lim_{x\to x_0} \frac{dn_\alpha}{dx}(x) = \frac{1}{u_\alpha(x_0)} \left\{ g_\alpha(n_e(x_0)) - n_\alpha(x_0) \lim_{x\to x_0} \frac{du_\alpha}{dx}(x) \right\} = -\infty.$$

We thus have proved the properties (iii)-(v) of the theorem.

We pass to the second step of the proof. It remains to show the existence of a (physically admissible) solution in a neighborhood of $x = 0$ which satisfies the properties (i) and (ii) of the theorem. This is far from being obvious since the differential system (3.4) is singular at $x = 0$. We begin by constructing an approximate solution of (2.13)-(2.17) in the following way. For any $\eta > 0$ arbitrarily small and for $1 \le \alpha \le p$, we introduce an approximation g_α^η of g_α satisfying the properties

$$g_\alpha^\eta(n_e) = g_\alpha(1)\ , \quad 1 - \eta \le n_e \le 1, \tag{5.4}$$

$$g_\alpha^\eta \quad \text{is bounded in } W^{1,\infty}(0,1), \tag{5.5}$$

$$g_\alpha^\eta \to g_\alpha \quad \text{in } C^0([0,1)) \quad \text{as } \eta \to 0. \tag{5.6}$$

This is indeed possible since we have assumed $g_\alpha \in C^1([0,1])$. We then solve the problem (2.13)-(2.17) with g_α replaced by g_α^η. We know from Example 2 that such a problem has a unique solution as long as the corresponding electron density n_e^η satisfies $n_e^\eta \ge 1 - \eta$. At the point a^η such that

$$n_e^\eta(a^\eta) = 1 - \eta,$$

the first part of the proof ensures that we can extend the solution up to a point x_0^η with.

$$n_e^\eta(x_0^\eta) \le \frac{1}{2}.$$

Let $\boldsymbol{U}^\eta$ be the solution thus obtained; we want to prove that $\boldsymbol{U}^\eta$, or at least a subsequence, converges towards a solution of (3.4),(3.5), or (2.13)-(2.17), as η tends to zero.

First, we show that $\boldsymbol{U}^\eta$ is defined in a fixed interval $[0, x_1]$ independent of η. Indeed, using the first inequality (4.14), we have

$$n_e^\eta(x) \ge 1 - 4g^\eta(1)^2x^2 = 1 - 4g(1)^2x^2.$$

Hence for

$$1 - 4g(1)^2x^2 \ge \frac{1}{2},$$

i.e., for

$$x \le \frac{1}{2\sqrt{2}g(1)},$$

we have

$$n_e^\eta(x) \geq \frac{1}{2}.$$

The approximate solution $\boldsymbol{U}^{\boldsymbol{\eta}}$ is thus defined in an interval containing $\left[0, \dfrac{1}{2\sqrt{2}g(1)}\right]$ and we can choose for x_1 any number less than $\dfrac{1}{2\sqrt{2}g(1)}$.

Let us next show that one can extract from (n_e^η) a subsequence which converges towards a solution of (5.2). Choosing x_1 small enough, we deduce from Lemmas 4, 6 and 7 that n_e^η remains in a bounded set of $W^{2,\infty}(0, x_1)$ as η tends to zero. Hence we can extract a subsequence still denoted by (n_e^η) such that

$$n_e^\eta \longrightarrow n_e \quad \text{in } C^1([0, x_1]).$$

By passing to the limit, we obtain that n_e is a decreasing function solution of (5.2) which satisfies

$$\begin{aligned} &n_e(0) = 1, \\ &\frac{1}{2} \leq n_e(x) \leq 1 \quad \text{for all } x \in [0, x_1] \end{aligned}$$

together with the estimates (4.14) and (4.22). In addition, j_α defined from n_e by (4.1) satisfies the estimates (4.16) while k_α defined from n_e by (5.1) satisfies the estimates (4.15). Setting

$$n_\alpha = \frac{j_\alpha^2}{k_\alpha} = \frac{j_\alpha^2}{\sqrt{-2\int_0^x (\frac{j_\alpha^2}{n_e}\frac{dn_e}{dx})(y)dy}}.$$

we find easily that $\boldsymbol{U}$ is a solution of the problem (3.4), (3.5) in $[0, x_1]$ or equivalently $\{(n_\alpha, u_\alpha); 1 \leq \alpha \leq p\}$ is a solution of the equations (2.13)-(2.17).

Let us then check that $\boldsymbol{U}$ is a physically admissible solution. Since g_α is assumed to be a C^1 function, j_α a C^2 function. Moreover we have

$$j_\alpha(x) > 0 \quad \text{for all } x > 0.$$

Next it follows from (5.1) and (4.15) that k_α is a C^1 function which satisfies

$$\begin{cases} k_\alpha(0) = \dfrac{dk_\alpha}{dx}(0) = 0, \\ \\ k_\alpha(x) > 0 \quad \text{for all } x > 0. \end{cases}$$

Then n_α is clearly a C^1 function for $x > 0$ and

$$n_\alpha(x) > 0 \quad \text{for all } x > 0.$$

It remains to analyze the behavior of n_α at $x = 0$. We have by (5.5)

$$|g_\alpha(n_e) - g_\alpha(1)| \leq C_1|n_e - 1| \leq C_2 x^2$$

and therefore

$$j_\alpha(x) = g_\alpha(1)x + O(x^3).$$

Next, using (5.1), we have

$$k_\alpha^2(x) = -2\int_0^x \{g_\alpha(1)^2 y^2 + O(y^4)\}\left(\frac{1}{n_e}\frac{dn_e}{dx}\right)(y)dy =$$

$$= g_\alpha(1)^2 h(x) + O(x^6).$$

which gives since $h(x) = O(x^4)$

$$k_\alpha = g_\alpha(1)\sqrt{h} + O(x^4).$$

We thus obtain

$$n_\alpha = \frac{j_\alpha^2}{k_\alpha} = \frac{g_\alpha(1)^2 x^2 + O(x^4)}{g_\alpha(1)\sqrt{h} + O(x^4)} = g_\alpha(1)\frac{x^2}{\sqrt{h}} + O(x^2)$$

and

$$n_e = \sum_{\alpha=1}^{p} n_\alpha = g(1)\frac{x^2}{\sqrt{h}} + O(x^2) = 1 + O(x^2).$$

This yields

$$\sqrt{h} = g(1)x^2 + O(x^4) \tag{5.7}$$

and

$$n_\alpha = \frac{g_\alpha(1)}{g(1)} + O(x^2).$$

Hence n_α is continuously differentiable at $x = 0$ and we have (3.11).

It remains only to prove that n_e is a C^2 function. We use (4.23)-(4.25). Since A and B are C^1 functions for $x > 0$ and

$$B = \sum_{\alpha=1}^{p} \frac{j_\alpha^4}{k_\alpha^3} - n_e = \sum_{\alpha=1}^{p} \frac{n_\alpha}{u_\alpha^2} - n_e$$

is positive in $[0, x_1]$ by Lemma 1, we obtain that $\dfrac{dn_e}{dx}$ is a C^1 function for $x > 0$. On the other hand, using (4.11) and (5.7) we have

$$1 - n_e = \sum_{\alpha=1}^{p} g_\alpha(1)\sqrt{h} + O(x^4) = g(1)^2 x^2 + O(x^4).$$

Hence n_e is twice continuously differentiable at $x = 0$ and we have (3.12). This concludes the proof of Theorem 2. $\square$

6. Numerical Results

We present in this section numerical simulations corresponding to various ionization rates and various numbers of ion species. We also compare the results provided by a one-velocity fluid model and a kinetic model.

We first describe the numerical method of solution of the plasma approximation equations (2.14)-(2.17). It consists in solving the plasma equation (5.2). Since $n_e(x)$ is a strictly decreasing function on $[0, x_0[$, we can look equivalently for the inverse function $x(n_e)$ for $n_e \leq 1$. We write

$$n_e = \sum_{\alpha=1}^{p} \frac{j_\alpha^2(n_e)}{k_\alpha(n_e)}. \tag{6.1}$$

As $n_e(0) = 1$, we have by (4.1)

$$j_\alpha(n_e) = -\int_{n_e}^{1} g_\alpha(n) \frac{dx(n)}{dn_e} dn, \tag{6.2}$$

and by (5.1)

$$k_\alpha(n_e) = \sqrt{2 \int_{n_e}^{1} \frac{j_\alpha^2(n)}{n} dn}. \tag{6.3}$$

Note also that the Theorem 2 yields the following behaviors for n_e, j_α and k_α as $x \to 0$:

$$n_e(x) \approx 1 - g(1)^2 x^2,\ j_\alpha(x) \approx g_\alpha(1) x,\ k_\alpha(x) \approx g_\alpha(1) g(1) x^2.$$

Hence, we obtain as $n_e \to 1$

$$x(n_e) \approx \frac{1}{g(1)} \sqrt{1 - n_e},\ j_\alpha(n_e) \approx \frac{g_\alpha(1)}{g(1)} \sqrt{1 - n_e},\ k_\alpha(n_e) \approx g(1) g_\alpha(1)(1 - n_e). \tag{6.4}$$

In order to solve the equations (6.1)-(6.3), we introduce a mesh in the variable n_e

$$n_e^0 = 1 > n_e^1 > n_e^2 > \ldots > n_e^i > n_e^{i+1} > \ldots$$

and we set

$$\Delta n^i = n_e^i - n_e^{i+1}.$$

We denote by x^i, j_α^i and k_α^i the approximate values of $x(n_e^i)$, $j_\alpha(n_e^i)$ and $k_\alpha(n_e^i)$ respectively. These sequences are initialized by

$$n^0 = 1,\ x^0 = 0,\ j_\alpha^0 = 0,\ k_\alpha^0 = 0.$$

and for the first index $i = 1$, using the expansion (6.4) of the solution near the origin, we take

$$x^1 = \frac{1}{g(1)} \sqrt{\Delta n^0},\ \ j_\alpha^1 = \frac{g_\alpha(1)}{g(1)} \sqrt{\Delta n^0},\ \ k_\alpha(x) = g_\alpha(1) g(1) \Delta n^0. \tag{6.5}$$

For $i \geq 1$, we compute x^{i+1}, j_α^{i+1} and k_α^{i+1} from x^i, j_α^i and k_α^i by

$$\begin{cases} x^{i+1} = x^i + \Delta n^i z^i, \\ j_\alpha^{i+1} = j_\alpha^i - \Delta n^i g_\alpha(n_e^i) z^i \\ k_\alpha^{i+1} = \sqrt{(k_\alpha^i)^2 + \frac{2\Delta n^i}{n_e^i}(j_\alpha^i)^2} \end{cases} \tag{6.6}$$

where $z^i \simeq x'(n_e^i)$ is solution of the equation

$$n_e^{i+1} = \sum_{\alpha=1}^{p} \frac{(j_\alpha^{i+1})^2}{k_\alpha^{i+1}} \tag{6.7}$$

Eq. (6.7) is indeed an equation of the second degree in z^i. Let us now analyze its solvability. For the sake of simplicity, we drop the indices i and we put a tilde on the quantities corresponding to the index $i+1$. Then, taking into account the 2nd equation (6.6), we obtain that z satisfies

$$(\Delta n)^2 \sum_{\alpha=1}^{p} \frac{g_\alpha^2(n_e)}{\tilde{k}_\alpha} z^2 - 2\Delta n \sum_{\alpha=1}^{p} \frac{j_\alpha g_\alpha(n_e)}{\tilde{k}_\alpha} z + \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{\tilde{k}_\alpha} - n_e + \Delta n = 0 \tag{6.8}$$

Since $x'(n_e) < 0$, we look for the negative root (if it exists) of (6.8). The sum of the roots being positive, such a negative root exists if and only if

$$\sum_{\alpha=1}^{p} \frac{j_\alpha^2}{\tilde{k}_\alpha} - n_e + \Delta n < 0.$$

Since

$$n_e = \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha},$$

the above condition reads

$$\Delta n < H(\Delta n) \tag{6.9}$$

where

$$H(\xi) = \sum_{\alpha=1}^{p} \frac{j_\alpha^2}{k_\alpha}(1 - (1 + 2\frac{j_\alpha^2 \xi}{n_e k_\alpha^2})^{-1/2}). \tag{6.10}$$

Note that $H(0) = 0$ and H' is a strictly decreasing function on $\mathbf{R}^+$ while H'' is a strictly increasing one. Hence,

$$\Delta n H'(0) + \frac{(\Delta n)^2}{2} H''(0) \leq H(\Delta n) \leq \Delta n H'(0). \tag{6.11}$$

First, using the 2nd inequality of (6.11), we obtain that

$$H'(0) \geq 1,$$

is a necessary condition for (6.9) to hold. This condition reads

$$\frac{1}{n_e}\sum_{\alpha=1}^{p}\frac{j_\alpha^4}{k_\alpha^3} \geq 1,$$

or equivalently

$$\sum_{\alpha=1}^{p}\frac{n_\alpha}{u_\alpha^2} \geq n_e.$$

This means that $x(n_e)$ belongs indeed to the interval of existence $[0, x_0]$ of the plasma approximation. Next, using the first inequality of (6.11), we find that

$$\Delta n \leq \Delta n H'(0) + \frac{(\Delta n)^2}{2} H''(0),$$

is a sufficient condition for (6.9) to hold. It reads

$$\Delta n \leq \frac{2n_e}{3}\,\frac{\sum_{\alpha=1}^{p}\frac{j_\alpha^4}{k_\alpha^3} - n_e}{\sum_{\alpha=1}^{p}\frac{j_\alpha^6}{k_\alpha^5}} \tag{6.12}$$

and provides an upper bound for Δn at each step of the computation. The discriminant of the second order equation 6.8 reads

$$\Delta = 4(\Delta n)^2\left\{\left(\sum_{\alpha=1}^{p}\frac{j_\alpha g_\alpha}{\sqrt{\tilde{k}_\alpha}}\right)^2 - \left(\sum_{\alpha=1}^{p}\frac{g_\alpha^2}{\sqrt{\tilde{k}_\alpha}}\right)\left(\sum_{\alpha=1}^{p}\frac{j_\alpha^2}{\sqrt{\tilde{k}_\alpha}} - n_e + \Delta n\right)\right\}$$

and it is obviously positive according to (6.12). The other physical quantities (partial density and velocity for the α-th species) are then obtained easily by

$$n_\alpha^i = (j_\alpha^i)^2/k_\alpha^i,\ u_\alpha^i = k_\alpha^i/j_\alpha^i. \tag{6.13}$$

Let us first present numerical results corresponding to a hydrogen-helium plasma, in equal part, with two ion species H_2^+ and H_e^+. We assume that the ionization process is driven by primary electrons of high energy ($1keV$) and by secondary electrons of low energy ($16eV$) which have been produced through the ionization of neutrals by primary electrons and then heated. The case $\gamma = 0$ corresponds to ionization by primary electrons while the case $\gamma = 1$ corresponds to ionization by the secondary electrons as in the definition of the ionization rates g. The ionization coefficients taken from [7] are given by

$$\begin{array}{lcl} H_2^+ & : & 3.85\ 10^{-8}cm^3s^{-1}(primary),\ \ 1.66\ 10^{-8}cm^3s^{-1}(secondary) \\ H_e^+ & : & 2.4\ 10^{-8}cm^3s^{-1}(primary),\ \ 3.6\ 10^{-9}cm^3s^{-1}(secondary) \end{array}$$

and are related to the cross section σ and the velocity V of the electrons (defined from their energy $1keV$ or $16eV$) by

$$S = \sigma V.$$

Since the densities n_a of the target particle in the ionization process (hydrogen and helium) are supposed constants and equal, the collision frequencies ν in the definition of g are proportionnal to the above ionization coefficients S

$$\nu = n_a S.$$

Thus, the *scaled* ionization rates are of the form:
case 1: $p = 2, \quad g_1(n_e) = 2.4 + 0.36 n_e, \quad g_2(n_e) = 3.85 + 1.66 n_e.$

We have plotted the following quantities: ion densities n_α and electron density n_e (Fig. 1), ion velocities u_α (Fig. 2) and current densities j_α and j (Fig. 3).

In fact, we are interested more specifically by the values of the physical quantities at the point x_0 (characterized in Theorem 2) since these values are to be taken as injection parameters for an ion beam extracted from the neutral plasma. We obtain

$$\begin{aligned} n_1 &= 0.1695, \quad & j_1 &= 0.1690, \quad & u_1 &= 0.9974, \\ n_2 &= 0.3304, \quad & j_2 &= 0.3309, \quad & u_2 &= 1.0013, \\ n_e &= 0.499996, \quad & j &= 0.4999987. \end{aligned}$$

Observe that the nondimensional velocity u_α of each species is close to 1 and moreover

$$n_e(x_0) \approx \frac{1}{2}, \quad j(x_0) \approx \frac{1}{2}. \tag{6.14}$$

Hence, the total extracted current density is approximatively given by Bohm's criterion as in the single ion species case considered in Example 1. Somewhat surprisingly, this seems to be a general feature of this model, i.e., (6.14) holds true. For illustrating this unexplained property, we have considered three cases (which have not necessarily a physical meaning) corresponding to highly different numbers of species or ionization rates:

(i) *case 2*: $p = 2$, $g_1(n) = 1$, $g_2(n) = n$.

(ii) *case 3*: $p = 2$, $g_1(n) = 1$, $g_2(n) = 10^6\, n^2$,

(iii) *case 4*: $p = 10$, $g_\alpha(n) = C_\alpha n^\alpha$, $\alpha = 1, \cdots, 10$, with the following arbitrary values for the C_α:
$$1, 1, 0.5, 3, 1, 0.25, 8, 0.5, 3, 100.$$

We give the values of the (scaled) electronic density and of the extracted current at x_0

Case	x_0	n_e / u_1	j / u_2	j_1 / n_1	j_2 / n_2
2	2.6514 10^{-01}	4.998 10^{-01} 9.8766 10^{-01}	4.9995 10^{-01} 1.01490 10^{+00}	2.6514 10^{-01} 2.6845 10^{-01}	2.3481 10^{-01} 2.31363 10^{-01}
3	6.7611 10^{-07}	4.9002 10^{-01} 9.3237 10^{-01}	4.9002 10^{-01} 1.0000 10^{+00}	6.7611 10^{-07} 7.2515 10^{-07}	4.90021 10^{-01} 4.9002 10^{-01}
4	1.9419 10^{-02}	4.9086 10^{-01} 7.6288 10^{-01}	4.9803 10^{-01} 8.0728 10^{-01}	1.9419 10^{-02} 2.5455 10^{-02}	1.44308 10^{-03} 1.7875 10^{-02}

In the case 4, we give below the partial currents, velocities and densities for the ten species

j_α (1.9420 10^{-02}, 1.4430 10^{-02}, 5.5456 10^{-03}, 2.640 10^{-02}, 7.1979 10^{-03}, 1.5110 10^{-03},4.1559 10^{-02}, 2.2775 10^{-03}, 1.2187 10^{-02}, 3.6749 10^{-01}),

u_α (7.6288 10^{-01}, 8.0728 10^{-01}, 8.5101 10^{-01}, 8.9260 10^{-01}, 9.3089 10^{-01}, 9.6513 10^{-01},9.9504 10^{-01}, 1.0207 10^{+00}, 1.0424 10^{+00}, 1.0607 10^{+00})

n_α (2.5455 10^{-02}, 1.7875 10^{-02}, 6.5165 10^{-03}, 2.9587 10^{-02}, 7.7322 10^{-03} , 1.5656 10^{-03}, 4.1766 10^{-02}, 2.2313 10^{-03}, 1.1691 10^{-02}, 3.4644 10^{-01}).

Let us now compare the results obtained by using this fluid model with those obtained using two other fluid and kinetic related models. As a reference case, we will choose the case 2 considered above.

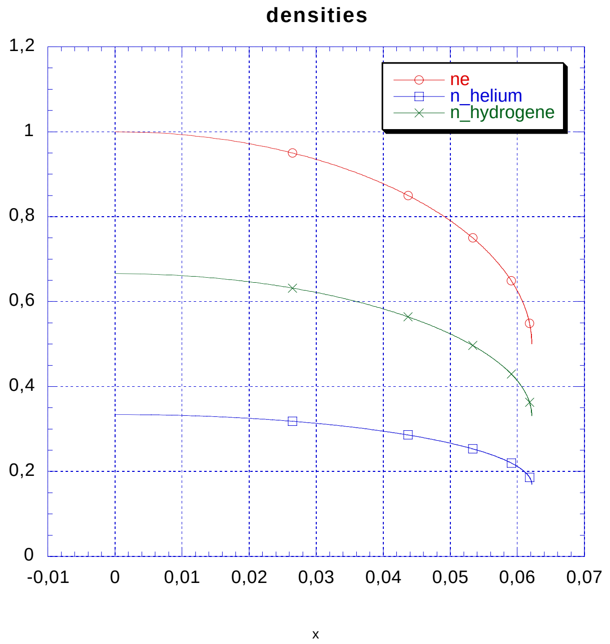

Fig. 1. *Case* $H^{+} - H_e$ *- densities*

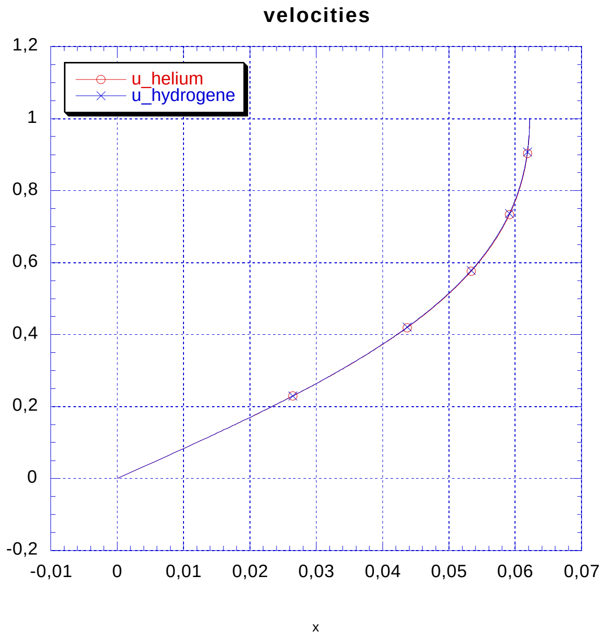

Fig. 2. *Case* $H^{+} - H_e$ *- velocities*

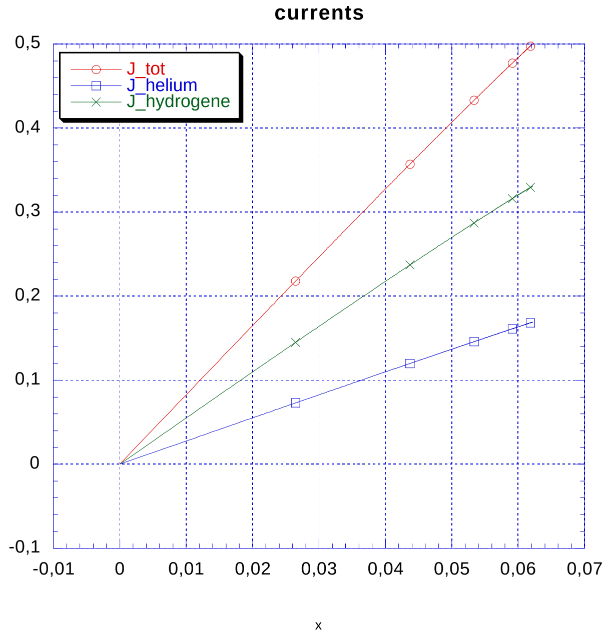

Fig. 3. *Case* $H^+ - H_e$ *- currents*

a) *A one-velocity fluid model.* We have already observed that the *scaled* velocities u_α are approximately equal [†]Hence, it makes sense to consider the following one-velocity plane model under the same physical hypotheses (stationarity, cold ions)

$$\begin{cases} \frac{d(n_\alpha u)}{dx} = g_\alpha(n_e), \\ \frac{d(n_\alpha u^2)}{dx} + \frac{n_\alpha}{n_e}\frac{dn_e}{dx} = 0, \\ n_e(0) = 1, u(0) = 1. \end{cases} \tag{6.15}$$

We note that the pair (n_e, u) is indeed the solution of the model (2.16) considered in Example 1. Thus, we have:

$$n_e(x_0) = 1/2, \quad j(x_0) = 1/2.$$

In the case 2, i.e., $p = 2$, $g_1 = 1$ and $g_2 = n$, the function $x(n)$ can be determined explicitly

$$x(n) = \frac{\pi}{2} - \sin^{-1}(2n-1) + \frac{3\sqrt{2}}{4}\left(\sin^{-1}\left(\frac{3n-1}{n+1}\right) - \frac{\pi}{2}\right)$$

which gives the following values

$$\begin{cases} n_1 = j_1 = \int_0^{x_0} g_1(n(y))dy = x_0 = x(1/2) = \frac{\pi}{2} + \frac{3\sqrt{2}}{4}\left(\sin^{-1}(\frac{1}{3}) - \frac{\pi}{2}\right) \approx 0.265.., \\ n_2 = j_2 = j - j_1 \approx 0.235. \end{cases}$$

which are very close to those provided by the multifluid model.

[†]Note however that the physical velocities $U_\alpha = \sqrt{\frac{Z_\alpha k_B T_e}{m_\alpha}} u_\alpha$ are all distinct.

b) *A multi-species kinetic model.* Let us now compare the above results with the numerical values given by a kinetic model. In the multispecies kinetic model introduced in [6], the fluid equations (1.8)-(1.9) are replaced by the Vlasov equation with ionization source term

$$v\frac{\partial f_\alpha}{\partial x} + \frac{d\varphi}{dx}\frac{\partial f_\alpha}{\partial v} = g_\alpha(n_e)\delta(v)\ ,\ x > 0\ ,\ v \in \mathbf{R} \tag{6.16}$$

where f_α is the distribution function of the ion species α and the boundary conditions (1.11) by

$$f_\alpha(0, v) = 0\ ,\ v > 0. \tag{6.17}$$

Then, we look for functions f_α, $1 \leq \alpha \leq p$ and φ solutions of (1.7)-(6.16)-(1.10) with the boundary conditions (6.17) and (1.12) where in the Poisson equation (1.10) we have

$$n_\alpha = \int_{-\infty}^{\infty} f_\alpha dv.$$

Now, it is a simple matter to generalize the results of [1] which consider the case of a single ion species to the case of several ion species as explained in [6]. Assuming that the function φ is monotone increasing, we obtain

$$n_\alpha(x) = \int_0^x \frac{g(\exp(-\varphi(y)))dy}{\sqrt{2\left(\varphi(x) - \varphi(y)\right)}},$$

so that the plasma approximation $\sum_{\alpha=1}^p n_\alpha = n_e = \exp(-\varphi)$ amounts to find φ solution of

$$\int_0^x \frac{g(\exp(\varphi(y))}{\sqrt{2(\varphi(x) - \varphi(y))}} dy = \exp(-\varphi(x)),\ \varphi(0) = 0. \tag{6.18}$$

Again, the plasma equation (6.18) has a solution φ defined in a maximal interval $[0, x_0)$ with $x_0 < +\infty$ and we find

$$\varphi(x_0) \approx 0.854,\ \ n(x_0) \approx 0.426,\ \ j(x_0) \approx 0.486.$$

independently of the ionization rates g_α. Moreover, in the plasma approximation, choosing (φ, v) instead of (x, v) as the set of independent variables, we obtain

$$f_\alpha(\varphi, v) = \begin{cases} \frac{\sqrt{2}}{\pi}\,\frac{g_\alpha(\exp(\frac{v^2}{2}-\varphi))}{g\exp((\frac{v^2}{2}-\varphi)}\left[\frac{1}{\sqrt{\varphi-\frac{v^2}{2}}} - 2F(\sqrt{\varphi - \frac{v^2}{2}})\right], 0 < v < \sqrt{2\varphi} \\ \\ 0 \ \text{ otherwise.} \end{cases} \tag{6.19}$$

where F is given by $F(x) = \exp(-x^2)\int_0^x \exp(t^2)dt$. In addition, the function $x(\varphi)$ can be computed by integrating the following ordinary differential equation

$$x'(\varphi) = \frac{2\sqrt{2}}{\pi g(\exp(-\varphi))}\frac{d}{d\varphi}F(\sqrt{\varphi}).$$

In the case 2 where $g_1 = 1$, $g_2 = n$, we find that the partial extracted currents are

$$j_1 = x_0 = 0.261116, \;\; j_2 = j_\infty - j_1 = 0.225995.$$

Note also that the scaled temperature is equal to 0.046. This small value of the temperature of the solution of the kinetic model is a justification of the hypothesis of cold ions used in the other models.

Let us summarize the density and current at the end of the plasma sheath for each model:

Model	$n(x_0)$	$j(x_0)$	$j_1 = x_0$	$j_2 = j_(x_0) - j_1$
1: multifluid (2.13)-(2.17)	0.499	0.499	0.265	0.235
2: fluid , one velocity (6.15)	0.5*	0.5*	0.265	0.235
3: kinetic ()	0.426*	0.486*	0.261	0.226

The quantities with $\star$ are independent of the ionization rates. Note that the results are similar. In particular, the ratio j_1/j_2 is close to 1.1 in all the models.

7. Extensions to more complex models

Let us mention some possible extensions of the present analysis. First, it is possible to take into account additional source or friction terms (section 6.1). It should be also interesting to perform a asymptotic analysis of the quasineutral system ($\varepsilon \to 0$) i.e. to justify from the mathematical point of view the convergence of solutions of system (1.7)-(1.10) toward solutions of system (1.14)-(1.17) we have constructed. Finally, let us point out that the hyperbolicity of the associated evolution problem imposes that the ion velocities remains equal (section 6.2).

7.1. *Friction terms*

We can include friction terms in our quasineutral model without changing the conclusions of Theorem 3. Consider for simplicity the case of a single ion species studied in Example 2. If we replace the second equation (2.16) by

$$\frac{d}{dx}(nu^2 + n) = -f(n,u),$$

where $f(n,u)$ stands for a friction term which satisfies:

$$f(n,u) \geq 0, \;\; \forall n \geq 0, \;\; \forall u \in \mathbf{R},$$

we obtain again that the modified system has a unique solution defined on a maximal interval $[0, x_0)$, $x_0 < +\infty$, and moreover, we have

$$u(x_0) = 1, \;\; n(x_0) \leq 1/2.$$

The proof goes along the same lines, but in that simple case in a much simpler way than the proof of Theorem 2.

7.2. *Evolution problems.*

Consider the nonstationary problem associated with the quasineutral model $(1.13)-(1.15)$. It can be written in the form

$$\begin{cases} \frac{dn_\alpha}{dt} + \frac{d(n_\alpha u_\alpha)}{dx} = g_\alpha(n_e), \;\; n_e = \sum_{\alpha=1}^{p} n_\alpha, \;\; 1 \leq \alpha \leq p, \\ \frac{du_\alpha}{dt} + u_\alpha \frac{du_\alpha}{dx} + \frac{T_e}{n_e}\frac{dn_e}{dx} = -g_\alpha(n_e) u_\alpha / n_\alpha, \; 1 \leq \alpha \leq p. \end{cases} \tag{7.1}$$

This first order nonlinear system is clearly hyperbolic for $p = 1$. In the case $p = 2$, extending the results of [8], one can show that the system is hyperbolic if and only if $u_1 = u_2$. This suggests to pay attention to the one-velocity fluid model (6.15).

8. Conclusions

As we have seen, despite the fact that the model from the physical point of view is very rough, it is complicated from the mathematical point of view. From the physical point of view, the main approximation in this model is the Maxwell-Boltzmann assumption for the electrons and the mathematical validity of this assumption is an open problem.

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