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Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics

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Abstract

This work addresses some asymptotic regimes for the coupling of radiation and hydrodynamics, and is inspired by the still non-answered need of high resolution and robust schemes for the numerical solutions of these problems. Using a simple characterization of the isotropy of the scattering in the comobile reference frame, we derive various asymptotic regimes. Among them is the non-equilibrium regime. Then we prove that the method of moments is compatible with the non-equilibrium regime. We also study the Rankine–Hugoniot relations.

Keywords: Radiative hydrodynamics; Non-equilibrium diffusion regime; Method of moments

1. Introduction

This work addresses some asymptotic regimes for the coupling of radiation and hydrodynamics, and is inspired by the still non-answered need of high resolution and robust schemes for the numerical solutions of these problems. Following Lowrie et al. [1] and Lowrie and Morel [2], we consider that numerical progress should be possible using *Eulerian* conservative high-order Godunov-type scheme.

But it raises many difficulties: many models are written in the comobile or Lagrangian reference frame which moves with the fluid, see [3–9] and references therein. For mathematical aspects of the radiative transfer equation and related issues see for instance [10]. Actually, one can distinguish at least three approaches: a purely Lagrangian approach where everything is calculated in the moving reference frame; a comobile approach where some quantities are calculated in Eulerian reference frame and others are calculated in the moving reference frame (see for example [8]); the last approach

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is purely Eulerian, everything is calculated in the Eulerian reference frame. This last approach, purely Eulerian, is the one we address here. The main idea of this work is that these Eulerian models should at least contain non-equilibrium models, that is models where the temperature of the fluid is different from the temperature of the radiation $T \neq T_r$. To our knowledge, a direct derivation of non-equilibrium model in the Eulerian frame was still missing. This is one of the contribution of this work. The method used is a simplified asymptotic analysis expansion, very similar to a *Chapman–Enskog* or *Hilbert* expansion, with an appropriate scaling of the flow where the scattering source term is of course relativistic as prescribed in [3] and others. For the sake of simplicity of mathematical developments the analysis is made in the context of the gray hypothesis.

An important and original feature of our analysis is the compatibility of all the expansions with the mathematical structure of the scattering. A consequence is that we really need to do the analysis using v and $\vec{n}$, and thus we cannot pre-integrate along v as it can be done with absorption only.

The main trick is a formula which allows to simplify the algebraic complexity of the expansion: this formula is a simple characterization in the lab frame of the isotropy of the scattering in the comobile frame. The consequence is the appearance of the famous non-conservative $p_r\nabla.\vec{v}$ term in the radiation equation. Note that this point was already emphasized in [3] for example, but in a Lagrangian frame and with a lot physical intuition, and discussion of the various scales of the problem (a modern presentation is [11]). The proof we give is more mathematically based and uses invariance relations true for this class of Lorentz models as a main tool to simplify the analysis: in particular the invariance of the measure $v\,\mathrm{d}v\,\mathrm{d}\vec{n} = v_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$.

In a second part we study another class of models, the so-called moment models. We consider the most standard and simplest one with two unknowns which are E_r the energy of the radiation and $\vec{F}_r$ the radiation flux. Then, we prove that this moment model contains the non-equilibrium limit with the $p_r\nabla.\vec{v}$ for *smooth solutions*, using once more a Chapman–Enskog expansion. To our surprise it appears this is not true for *discontinuous solutions*, such as shocks and contact discontinuities: discontinuous solutions of the moment model do not tend to discontinuous solutions of the non-equilibrium model. To fix this question we propose a modification of the moment model, which is based on the choice of new unknowns which are S_r the entropy of the radiation and $\vec{F}_r$ the radiation flux. We prove that this modified moment model contains the non-equilibrium limit for both smooth solutions *and* discontinuous solutions. Since numerical methods based on Godunov methods and the Riemann problem need these discontinuous solutions, it is an indication that the modified moment model with $(S_r, \vec{F}_r)$ is better suited for the development of conservative and Eulerian numerical schemes than the classical moment model with $(E_r, \vec{F}_r)$. Numerical results will be presented in a future work.

2. Models

2.1. Relativistic gas dynamics

The Euler system of inviscid gas dynamics with full Lorentz invariance is [3,6,12]

$$\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,$$

$$\frac{\partial}{\partial t}\left(\gamma\left(1+\frac{h}{c^2}\right)\rho\vec{v}\right)+\nabla.\left(\gamma\left(1+\frac{h}{c^2}\right)\rho\vec{v}\otimes\vec{v}+p\mathbf{I}\right)=0,$$

$$\frac{\partial}{\partial t}\left(\gamma\rho\left(1+\frac{h}{c^2}\right)-\frac{p}{c^2}\right)+\nabla.\left(\gamma\rho\left(1+\frac{h}{c^2}\right)\vec{v}\right)=0. \tag{1}$$

For this kind of Lorentz invariant models, one must recall that there is a distinction between the Eulerian reference frame also referred to as the lab frame, and the comobile reference frame which moves with the fluid also referred to as the Lagrangian frame. The density $\rho=1/\tau$ in the lab frame is different from the density calculated in the comobile frame $\rho_0=1/\tau_0$. In what follows the subscript 0 will designate any quantity measured in the comobile frame. One has $\rho=\gamma\rho_0$ where γ is defined by

$$\gamma=\frac{1}{\sqrt{1-|\vec{v}|^2/c^2}} \tag{2}$$

where c is the velocity of light. In (1), h is the enthalpy of the fluid calculated in the comobile frame $h=e+p\tau_0$, where p is the pressure. If one assumes for simplicity a perfect gas pressure law

$$p=\Gamma\frac{e}{\tau_0}, \quad \Gamma>0, \tag{3}$$

then the enthalpy is simply $h=(\Gamma+1)e$. Here, e is the internal energy of the fluid calculated in the comobile frame. Multiplying the last equation of (1) with c^2 and subtracting the first one multiplied by c^2 for the sake of convenience, we rewrite it as

$$\frac{\partial}{\partial t}(\rho)+\nabla.(\rho\vec{v})=0,$$

$$\frac{\partial}{\partial t}\left(\gamma\left(1+\frac{h}{c^2}\right)\rho\vec{v}\right)+\nabla.\left(\gamma\left(1+\frac{h}{c^2}\right)\rho\vec{v}\otimes\vec{v}+p\mathbf{I}\right)=0,$$

$$\frac{\partial}{\partial t}\left(c^2\gamma\rho\left(1+\frac{e}{c^2}+\frac{|\vec{v}|^2}{c^2}\frac{p\tau_0}{c^2}\right)-c^2\rho\right)$$

$$+\nabla.\left(c^2\left(\gamma\rho\left(1+\frac{e}{c^2}+\frac{|\vec{v}|^2}{c^2}\frac{p\tau_0}{c^2}\right)-c^2\rho\right)\vec{v}+p\vec{v}\right)=0. \tag{4}$$

For the sake of simplicity of notations, we define

$$\vec{v}_2=\gamma\left(1+\frac{h}{c^2}\right)\vec{v} \quad\text{and}\quad E_2=c^2\gamma\left(1+\frac{e}{c^2}+\frac{u^2}{c^2}\frac{p\tau_0}{c^2}\right)-c^2.$$

With these notations (4) is equivalent to

$$\frac{\partial}{\partial t}(\rho)+\nabla.(\rho\vec{v})=0,$$

$$\frac{\partial}{\partial t}(\rho\vec{v}_2)+\nabla.(\rho\vec{v}_2\otimes\vec{v}+p\mathbf{I})=0,$$

$$\frac{\partial}{\partial t}(\rho E_2)+\nabla.(\rho E_2\vec{v}+p\vec{v})=0. \tag{5}$$

2.2. Galilean gas dynamics

The classic Euler system of inviscid gas dynamics with Galilean invariance is recovered as the limit of (6) when $|\vec{v}|/c \to 0$. We consider the regime

$$\varepsilon = \frac{|\vec{v}|}{c}, \quad \frac{e}{c^2} = O(\varepsilon^2). \tag{6}$$

Indeed one has

$$\vec{v}_2 = \vec{v} + O(\varepsilon^2) \quad \text{and} \quad E_2 = e + \tfrac{1}{2}\,|\vec{v}|^2 + O(\varepsilon^2). \tag{7}$$

Let us define the classical total energy $E = e + \frac{1}{2}|\vec{v}|^2$. Then the classic Euler system of inviscid gas dynamics

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ &\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I}) = 0, \\ &\frac{\partial}{\partial t}(\rho E) + \nabla.(\rho E\vec{v} + p\vec{v}) = 0, \end{aligned} \tag{8}$$

is recovered as the $O(\varepsilon^2)$ approximation of (1). Thus, it is possible to base the coupling of radiation and hydrodynamics on (8) even if relaxation source terms must be Lorentz invariant for the final model to be correct. In order to simplify the presentation and since we are interested mainly in flows moving at moderate velocities, we use (8) instead of (1) in the rest of this paper.

2.3. Transfer equation for photons

The transfer equation for photons is

$$\frac{1}{c}\frac{\partial}{\partial t} I + \vec{n}.\nabla I = S_{\rm t}(\nu, \vec{n}), \tag{9}$$

where $I(t, x : \nu, n)$ is the intensity of the radiation, ν the frequency and $\vec{n}$ the direction of the photons. The source term is $S_{\rm t}(\nu, \vec{n})$. It is well known that the source term has to be Lorentz invariant for the total coupled system to be accurate [3] (but the derivation of the non-equilibrium diffusion limit that we give in this work is another proof that $S_{\rm t}(\nu, \vec{n})$ must be Lorentz invariant even for small velocities). In this work we follow [1] and consider a simplified source term where $S_{\rm t} = S_{\rm a} + S_{\rm s}$ is the sum of two contributions. The first one takes into account the absorption/re-emission of photons by the matter

$$S_{\rm a}(\nu, \vec{n}) = \frac{\nu_0}{\nu}\,\sigma_{\rm a}(\nu_0)\left[\left(\frac{\nu}{\nu_0}\right)^3 B(\nu_0, T) - I\right]. \tag{10}$$

Here $B(\nu_0, T)$ is the Planckian

$$B(\nu_0, T) = \frac{2h\nu_0^3}{c^2}(e^{h\nu_0/kT} - 1)^{-1} \tag{11}$$

and $\sigma_a(\nu_0) \geqslant 0$ is the absorption coefficient. In definitions (10)–(17), one has to use the frequency and direction of the photon calculated in the comobile frame

$$\nu_0 = \gamma\nu\left(1 - \frac{\vec{n}.\vec{v}}{c}\right) \quad \text{and} \quad \vec{n}_0 = \left(\frac{\nu}{\nu_0}\right)\left[\vec{n} - \frac{\gamma}{c}\,\vec{v}\left(1 - \frac{\vec{n}.\vec{v}}{c}\left(\frac{\gamma}{\gamma+1}\right)\right)\right]. \tag{12}$$

Another important invariance relation [3,6,8] between the intensity of the radiation in the lab frame and the intensity of the radiation in the comobile frame is

$$\frac{I}{\nu^3} = \frac{I_0}{\nu_0^3}. \tag{13}$$

Defining also the Planckian measured in the comobile frame as $B(\nu_0,T)/\nu^3 = B_0(\nu_0,T)/\nu_0^3$, one gets another expression of the absorption/re-emission contribution (10)

$$S_a(\nu,\vec{n}) = \frac{\nu^2}{\nu_0^2}\,\sigma_a(\nu_0)[B_0(\nu_0,T) - I_0(\nu_0,\omega_0)]. \tag{14}$$

The second term takes into account the scattering of photons by the matter [3]

$$S_s(\nu,\vec{n}) = \frac{\nu^2}{\nu_0^2}(S_s)_0(\nu_0,\vec{n}_0), \tag{15}$$

where the scattering measured in the comobile frame is

$$(S_s)_0(\nu_0,\vec{n}_0) = \sigma_s(\nu_0)\left[\frac{1}{4\pi}\int I(\nu_0,\vec{n}_0')\,\mathrm{d}\vec{n}_0' - I_0(\nu_0,\vec{n}_0)\right]. \tag{16}$$

Here, $\sigma_s(\nu_0) \geqslant 0$ is the scattering coefficient. Since the scattering (16) is clearly isotropic in the comobile frame one gets

$$\int\int (S_s)_0(\nu_0,\vec{n}_0)\,\mathrm{d}\nu_0\,\mathrm{d}\vec{n}_0 = 0.$$

Another possibility for the scattering is [1]

$$S_s(\nu,\vec{n}) = \frac{\nu_0}{\nu}\,\sigma_s(\nu_0)\left[\left(\frac{\nu}{\nu_0}\right)^3 \frac{1}{4\pi}\int \frac{\nu_0}{\nu'}\,I(\nu',\vec{n}')\,\mathrm{d}\vec{n}' - I\right]. \tag{17}$$

With [1] we use the following definition for ν' which enters in the scattering contribution:

$$\nu' = \nu\,\frac{1 - \vec{n}.\vec{v}/c}{1 - \vec{n}'.\vec{v}/c}. \tag{18}$$

In Appendix A we prove that (17) is equal to (15). An important consequence of (15) is

Lemma 1 (Characterization of the isotropy of the scattering). *The isotropy of the scattering in the comobile frame is characterized by*

$$\int\int \frac{\nu_0}{\nu}\,S_s(\nu,\vec{n})\,\mathrm{d}\nu\,\mathrm{d}\vec{n} = 0. \tag{19}$$

This is due to

$$
\begin{aligned}
0 &= \int\!\!\int (S_\mathrm{s})_0(v_0,\vec{n}_0)\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0,\\
&= \int\!\!\int \frac{v_0^2}{v^2}(S_\mathrm{s})_0(v_0,\vec{n}_0)\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0 = \int\!\!\int \frac{v_0}{v}(S_\mathrm{s})(v_0,\vec{n}_0)\,\mathrm{d}v\,\mathrm{d}\vec{n},
\end{aligned}
$$

where we use the invariance of the integration measure $v\,\mathrm{d}v\,\mathrm{d}\vec{n} = v_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$, see [3]. Eq. (19) means that even if the scattering is isotropic in the comobile frame, then the scattering is non-isotropic in the lab frame.

2.4. The full system for the coupling of radiation and hydrodynamics

In order to couple radiation and hydrodynamics we need the influence of radiation on the matter. So we define

$$S_E = \int\!\!\int S_\mathrm{t}\,\mathrm{d}v\,\mathrm{d}\vec{n} \quad \text{and} \quad \vec{S}_F = \frac{1}{c}\int\!\!\int \vec{n}S_\mathrm{t}\,\mathrm{d}v\,\mathrm{d}\vec{n}. \tag{20}$$

Here, $S_E(\vec{S}_F)$ characterizes the energy (resp. impulse) exchange between the radiation and the matter. Following [3] we modify (8) and get the system that couples gas dynamics and the radiation

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I}) = -\vec{S}_F,\\
&\frac{\partial}{\partial t}(\rho E) + \nabla.(\rho E\vec{v} + p\vec{v}) = -S_E,\\
&\frac{1}{c}\frac{\partial}{\partial t}I(v,\vec{n}) + \vec{n}.\nabla I(v,\vec{n}) = S_t(v,\vec{n}) \quad \forall v,\vec{n}.
\end{aligned} \tag{21}
$$

Note that a consequence of (19) is that the integrated scattering on the right-hand side of the system is $\int\!\int S_\mathrm{s}\,\mathrm{d}v\,\mathrm{d}\vec{n}$ and is generally non-zero. Thus, this term contributes to the right-hand side for the impulse and the energy equations. It is only the integrated scattered radiation in the comobile frame which is zero $\int\!\int (S_\mathrm{s})_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0 = 0$. Concerning the integrated scattered radiation in the lab frame we only have (19): this formula will play an important role in the asymptotic analysis of the system (21) in order to derive the non-equilibrium diffusion model.

2.5. Thermodynamic compatibility of the model

The physically correct radiative entropy (see [13]) is

$$S_\mathrm{r} = -\frac{2k}{c^3}\int\!\!\int v^2[n\log n - (n+1)\log(n+1)]\,\mathrm{d}v\,\mathrm{d}\vec{n}, \tag{22}$$

where by definition

$$n = \frac{c^2}{2h}\frac{I}{v^3} = \frac{c^2}{2h}\frac{I_0}{v_0^3} \tag{23}$$

is a relativistic invariant. First and second variations of $S_{\rm r}$ with respect to I are given in

$$\mathrm{d}S_{\rm r} = -\frac{k}{ch}\int\!\!\int \frac{1}{\nu}\log\left(\frac{n}{n+1}\right)\,\mathrm{d}I\,\mathrm{d}\nu\,\mathrm{d}\vec{n} \tag{24}$$

and

$$\mathrm{d}^2 S_{\rm r} = -\frac{c^2 k}{2h^2}\int\!\!\int \frac{1}{\nu^4}\,\frac{1}{n(n+1)}\,\mathrm{d}I\,\mathrm{d}I'\,\mathrm{d}\nu\,\mathrm{d}\vec{n}. \tag{25}$$

From (25) it is clear that the entropy is strictly concave with respect to I. On the other hand, it is clear that $S_{\rm r}$ given in (22) is not Lorentz invariant (only n and $\nu\,\mathrm{d}\nu\,\mathrm{d}\vec{n}$ are invariant). Let us define the radiative entropy flux $\vec{Q}_{\rm r}$:

$$\vec{Q}_{\rm r} = -\frac{2k}{c^2}\int\!\!\int \nu^2[n\log n - (n+1)\log(n+1)]\vec{n}\,\mathrm{d}\nu\,\mathrm{d}\vec{n}. \tag{26}$$

Consider a smooth solution of (21) where the right-hand side is $S_{\rm t} = S_{\rm a} + S_{\rm s}$ given in (10)–(17). Then one has

$$\partial_{\rm t} S_{\rm r} + \nabla\vec{Q}_{\rm r} = -\frac{k}{h}\int\!\!\int \frac{1}{\nu}\log\left(\frac{n}{n+1}\right)(S_{\rm a}+S_{\rm s})\,\mathrm{d}\nu\,\mathrm{d}\vec{n}. \tag{27}$$

Be careful that the notations for the source terms, $S_{\rm t}$, $S_{\rm a}$ and $S_{\rm s}$, is closed to the notations for the entropies, $S_{\rm r}$ and S. The entropy of the fluid is S with the fundamental law of thermodynamic

$$T\,\mathrm{d}S = \mathrm{d}\left(E - \frac{1}{2}\,\vec{v}.\vec{v}\right) + pd\,\frac{1}{\rho} = \mathrm{d}E - \vec{v}.d\vec{v} + pd\,\frac{1}{\rho}. \tag{28}$$

A standard consequence of (21) and (28) is

$$\partial_t(\rho S) + \nabla(\rho\vec{v}S) = \frac{1}{T}(-S_E + \vec{v}.\vec{S}_F), \tag{29}$$

that is

$$\begin{aligned}\partial_t(\rho S) + \nabla(\rho\vec{v}S) &= -\frac{1}{cT}\int\!\!\int (S_{\rm a}+S_{\rm s})\,\mathrm{d}\nu\,\mathrm{d}\vec{n} + \frac{\vec{v}}{cT}\int\!\!\int (S_{\rm a}+S_{\rm s})\vec{n}\,\mathrm{d}\nu\,\mathrm{d}\vec{n}\\ &= -\frac{1}{T}\int\!\!\int \left(1 - \frac{\vec{v}.\vec{n}}{c}\right)(S_{\rm a}+S_{\rm s})\,\mathrm{d}\nu\,\mathrm{d}\vec{n}.\end{aligned} \tag{30}$$

So the total entropy production is

$$\partial_t(\rho S + S_{\rm r}) + \nabla.(\rho\vec{v}S + \vec{Q}_{\rm r}) = \vec{Q}_{\rm a} + \vec{Q}_{\rm s}, \tag{31}$$

where

$$\vec{Q}_{\rm a} = -\frac{k}{h}\int\!\!\int \frac{1}{\nu}\log\left(\frac{n}{n+1}\right)S_{\rm a}\,\mathrm{d}\nu\,\mathrm{d}\vec{n} - \frac{1}{T}\int\!\!\int \left(1 - \frac{\vec{v}.\vec{n}}{c}\right)S_{\rm a}\,\mathrm{d}\nu\,\mathrm{d}\vec{n} \tag{32}$$

and

$$\vec{Q}_{\rm s} = -\frac{k}{h}\int\!\!\int \frac{1}{\nu}\log\left(\frac{n}{n+1}\right)S_{\rm s}\,\mathrm{d}\nu\,\mathrm{d}\vec{n} - \frac{1}{T}\int\!\!\int \left(1 - \frac{\vec{v}.\vec{n}}{c}\right)S_{\rm s}\,\mathrm{d}\nu\,\mathrm{d}\vec{n}. \tag{33}$$

Lemma 2 (Thermodynamic compatibility of the scattering). *The fluid entropy production due to the scattering is always zero* $\int\int(1-\vec{v}.\vec{n}/c)S_s\,\mathrm{d}v\,\mathrm{d}\vec{n}=0$. *The scattering entropy production* Q_s *is always non-negative* $\vec{Q}_s \geqslant 0$: $Q_s=0$ *if and only if the radiation is isotropic in the comobile frame.*

One has $\vec{Q}_s=\vec{Q}_s^{\mathrm{R}}+\vec{Q}_s^{\mathrm{F}}$ where $\vec{Q}_s^{\mathrm{R}}$ is the contribution of the transfer equation to the entropy production and Q_s^{F} is the fluid entropy production. First

$$\vec{Q}_s^{\mathrm{F}} = -\frac{1}{T}\int\int\left(1-\frac{\vec{v}.\vec{n}}{c}\right)(S_s)_0\,\mathrm{d}v\,\mathrm{d}\vec{n} = -\frac{1}{T}\int\int\frac{v^2}{v_0^2}\left(1-\frac{\vec{v}.\vec{n}}{c}\right)(S_s)_0\,\mathrm{d}v\,\mathrm{d}\vec{n}$$

$$=\frac{1}{\gamma T}\int\int\frac{v}{v_0}(S_s)_0\,\mathrm{d}v\,\mathrm{d}\vec{n}\quad \text{(see (12))}$$

$$=\frac{1}{\gamma T}\int\int(S_s)_0\,\mathrm{d}v\,\mathrm{d}\vec{n}\quad \text{(invariance of the measure).}$$

Due to the isotropy of the scattering in the comobile frame $\vec{Q}_s^{\mathrm{F}}=0$. The other term

$$\vec{Q}_s^{\mathrm{R}} = -\frac{k}{h}\int\int\frac{1}{v}\log\left(\frac{n}{n+1}\right)S_s\,\mathrm{d}v\,\mathrm{d}\vec{n} \tag{34}$$

with S_s is given by (15). So $\vec{Q}_s^{\mathrm{R}}=-k/h\int\int(1/v_0)\log(n/(n+1))\,(S_s)_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$ due to the Lorentz invariance of the measure $v\,\mathrm{d}v\,\mathrm{d}\vec{n}=v_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$. Then (16) implies $\vec{Q}_s^{\mathrm{R}}=\int\int(1/v_0)\sigma_s(v_0)q_s\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$, where

$$q_s=\left(f\left(\frac{1}{4\pi}\int I(v_0,\vec{n}_0)\,\mathrm{d}\vec{n}_0\right)-f(I_0)\right)\times\left(\frac{1}{4\pi}\int I(v_0,\vec{n}_0)\,\mathrm{d}\vec{n}_0)-I_0\right).$$

Here f denotes the function

$$f(x)=\frac{k}{h}\log\left(\frac{\frac{c^2}{2h}\frac{x}{v_0^3}}{\frac{c^2}{2h}\frac{x}{v_0^3}+1}\right) \tag{35}$$

such that $f(I_0)=(k/h)\log(n/(n+1))$, see Eq. (102). Since f is strictly increasing for non-negative x, then

$$q_s=f'\left((1-\alpha)I_0+\alpha\frac{1}{4\pi}\int I(v_0,\vec{n}_0)\,\mathrm{d}\vec{n}_0)\right)\left[\frac{1}{4\pi}\int I(v_0,\vec{n}_0)\,\mathrm{d}\vec{n}_0)-I_0\right]^2.$$

So $q_s\geqslant 0$ and is zero if and only if $(1/4\pi)\int I(v_0,\vec{n}_0)\,\mathrm{d}\vec{n}_0)-I_0=0$. Now the proof ends.

In order to state a similar result for the absorption–emission right-hand side S_a, a little difficulty arises. Indeed, we have replaced the Lorentz invariant Euler gas dynamics equation (4) by Galileo invariant gas dynamics equation (8), and this is a good approximation as soon as $|\vec{v}|/c$ is small. However, in complete rigor the source term in (21) should have been placed on the right-hand side of the relativistic gas dynamic system (4) but not on the right-hand side of Galilean gas dynamic system (8). The difference produces a small discrepancy which might give as well an eventually negative term in the entropy production. However, this term is proportional to $(|\vec{v}|/c)^2$ so it is meaningless for non-relativistic gases.

Lemma 3 (Thermodynamic compatibility of the absorption–emission). *The absorption–emission entropy production* $\vec{Q}_a$ *is the sum of two contributions* $\vec{Q}_a = P_a + \delta P_a$. *The first one is non-negative* $P_a \geqslant 0$ *and is zero if and only if the radiation intensity is equal to the Planckian in the comobile frame* $I_0 = B_0(v_0, T)$. *The correction* δP_a *is due to the approximation of the relativistic gas dynamics system by the Galilean gas dynamic system*: $\delta P_a = O(|\vec{v}|^2/c^2)$ *for smooth solutions.*

Let us go back to the definition of the entropy production $\vec{Q}_a$. We pose $\vec{Q}_a = \vec{Q}_a^R + \vec{Q}_a^F$ where $\vec{Q}_a^R$ (resp. $\vec{Q}_a^F$) is the first (resp. second) term in (32) and is the contribution of the radiation (resp. fluid) to the total entropy production.

From (31) one has

$$\begin{aligned}\vec{Q}_a^F &= \partial_i(\rho S) + \nabla(\rho \vec{v} S) = -\frac{1}{T}(S_E + \vec{S}_F \cdot \vec{v}) \\ &= -\frac{\gamma}{T}(S_E + \vec{S}_F \cdot \vec{v}) + \frac{\gamma - 1}{T}(S_E + \vec{S}_F \cdot \vec{v}) \\ &= -\frac{1}{T} \int\int \sigma_a S \gamma \left(1 - \frac{\vec{v}\cdot\vec{n}}{c}\right) \mathrm{d}v \, \mathrm{d}\vec{n} + \delta P_a,\end{aligned} \tag{36}$$

where by definition

$$\delta P_a = (\gamma - 1)\,\frac{S_E + \vec{S}_F \cdot \vec{v}}{T} = (\gamma - 1)[\partial_t(\rho S) + \nabla(\rho \vec{v} S)]. \tag{37}$$

δP_a is clearly second order since $\gamma - 1 = O(|\vec{v}|^2/c^2)$. Due to (14) and (13), one has

$$\vec{Q}_a^F = -\frac{1}{T} \int\int \sigma_a \frac{v^2}{v_0^2}(B_0(v_0, T) - I_0)\gamma \left(1 - \frac{\vec{v}\cdot\vec{n}}{c}\right) \mathrm{d}v \, \mathrm{d}\vec{n} + \delta P_a,$$

that is

$$\vec{Q}_a^F = -\frac{1}{T} \int\int \sigma_a (B_0(v_0, T) - I_0)\, \mathrm{d}v_0 \, \mathrm{d}\vec{n}_0 + \delta P_a \tag{38}$$

due to (12) and the invariance of the measure $v \, \mathrm{d}v \, \mathrm{d}\vec{n} = v_0 \, \mathrm{d}v_0 \, \mathrm{d}\vec{n}_0$. On the other hand, we have

$$\vec{Q}_a^R = -\frac{k}{h} \int\int \frac{\sigma_a}{v_0} \log\left(\frac{n}{n+1}\right) \frac{v^2}{v_0^2}\,(B_0(v_0, T) - I_0)\, \mathrm{d}v \, \mathrm{d}\vec{n},$$

that is

$$\vec{Q}_a^R = -\frac{k}{h} \int\int \frac{\sigma_a}{v_0} \log\left(\frac{n}{n+1}\right) (B_0(v_0, T) - I_0)\, \mathrm{d}v_0 \, \mathrm{d}\vec{n}_0. \tag{39}$$

Combining (38) and (39) we get $\vec{Q}_a = Q_a^R + Q_a^F = P_a + \delta P_a$ where by definition

$$\begin{aligned}P_a &= -\int\int \frac{\sigma_a}{v_0} \left(\frac{k}{h} \log\left(\frac{n}{n+1}\right) - \frac{v_0}{T}\right) (B_0(v_0, T) - I_0)\, \mathrm{d}v_0 \, \mathrm{d}\vec{n}_0 \\ &= \int\int \frac{\sigma_a}{v_0}\,(f(B_0(v_0, T)) - f(I_0))(B_0(v_0, T) - I_0)\, \mathrm{d}v_0 \, \mathrm{d}\vec{n}_0.\end{aligned}$$

The function f is defined in (35). So

$$P_a = \int\int \frac{\sigma_a(v_0)}{v_0} f'((1-\alpha)I_0 + \alpha B_0(v_0,T))[B_0(v_0,T) - I_0]^2 \, dv_0 \, d\vec{n}_0 \geqslant 0$$

and is zero if and only if $I_0 = B(v_0,T)$. Now, the proof is complete.

3. Simplified models

Following [1] we study some asymptotic regimes of the full system (21) by means of non-dimensional variables. First, we assume that σ_a and σ_s are independent of the frequency

$$\sigma_a(v_0) = \sigma_a, \quad \sigma_s(v_0) = \sigma_s. \tag{40}$$

This hypothesis is called the gray hypothesis. This is of course a very crude approximation, but is motivated by the mathematical analysis.

Second, we introduce some hydrodynamics scales where a_∞ is a characteristic value of the fluid velocity and so on

$$\begin{aligned} &x = \hat{x} l, \quad t = \hat{t} l / a_\infty, \quad \rho = \hat{\rho}\rho_\infty, \quad v = \hat{v} a_\infty, \\ &p = \hat{p}\rho_\infty a_\infty^2, \quad T = \hat{T} T_\infty, \quad v = \hat{v} k T_\infty / h, \quad I = \hat{I} h c a_r T_\infty^3 / k, \\ &\sigma_a = \hat{\sigma}_a / \lambda_a, \quad \sigma_s = \hat{\sigma}_s / \lambda_s. \end{aligned} \tag{41}$$

A caret denotes a non-dimensional quantity, and

$$a_r = \frac{8\pi^5 k^4}{15 c^3 h^3}. \tag{42}$$

We also define two non-dimensional parameters

$$\mathscr{C} = \frac{c}{a_\infty} \quad \text{and} \quad \mathscr{P} = \frac{a_r T_\infty^4}{\rho_\infty a_\infty^2}. \tag{43}$$

The first parameter, $\mathscr{C}$, is always large parameter for a flow non-relativistic. The second parameter, $\mathscr{P}$, measures the ratio of the radiative energy over the internal energy. It is possible to simplify by taking $\mathscr{P} = 1$ in many cases, but the sake of compatibility with the notations of [1] we keep $\mathscr{P}$. We refer the reader to the paper [1] where it is shown that the non-dimensional equations derived from (21) are

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ &\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I}) = -\mathscr{P}\vec{S}_F, \\ &\frac{\partial}{\partial t}(\rho E) + \nabla.(\rho E\vec{v} + p\vec{v}) = -\mathscr{P}\mathscr{C} S_E, \\ &\frac{1}{\mathscr{C}}\frac{\partial}{\partial t} I + \vec{n}.\nabla I = S_t(v,\vec{n}). \end{aligned} \tag{44}$$

The interaction is now characterized by $S_t = S_a + S_s$ with

$$S_a(v,\vec{n}) = \mathscr{L}\frac{v_0}{v}\,\sigma_a(v_0)\left[\left(\frac{v}{v_0}\right)^3 B(v_0,T) - I\right] \tag{45}$$

and

$$S_s(v,\vec{n}) = \mathscr{L}\mathscr{L}_s\,\frac{v_0}{v}\,\sigma_s(v_0)\left[\left(\frac{v}{v_0}\right)^3 \frac{1}{4\pi}\int \frac{v_0}{v'}\,I(v',\vec{n}')\,d\vec{n}' - I\right], \tag{46}$$

where

$$\mathscr{L} = \frac{1}{\lambda_a}, \quad \mathscr{L}_s = \frac{\lambda_a}{\lambda_s} \tag{47}$$

and

$$B(v_0,T) = \frac{15v_0^3}{4\pi^5}\,(e^{v_0/T} - 1)^{-1}, \tag{48}$$

$$v_0 = v\gamma_L(1 - \vec{n}.\vec{v}/\mathscr{C}), \quad v' = v\,\frac{1 - \vec{n}.\vec{v}/\mathscr{C}}{1 - \vec{n}'.\vec{v}/\mathscr{C}}, \tag{49}$$

$$\gamma_L = 1/\sqrt{1 - |\vec{v}|^2/\mathscr{C}^2}. \tag{50}$$

Since $\partial t E_r + \mathscr{C}\vec{F}_r = \mathscr{C}S_E$ we may prefer to use

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}\left(\rho\vec{v} + \frac{\mathscr{P}}{\mathscr{C}}\vec{F}_r\right) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I} + \mathscr{P}\mathbf{P}_r) = 0,\\
&\frac{\partial}{\partial t}(\rho E + \mathscr{P}E_r)\nabla.(\rho E\vec{v} + p\vec{v} + \mathscr{P}\mathscr{C}\vec{F}_r) = 0,\\
&\frac{1}{\mathscr{C}}f\,\frac{\partial}{\partial t}I + \vec{n}.\nabla I = S_t(v,\vec{n}),
\end{aligned} \tag{51}$$

instead of (44). Here, E_r (resp. $\vec{F}_r\mathbf{P}_r$) is simply the total integrated intensity (resp. flux, pressure tensor) of the radiation

$$E_r = \int\int I\,dv\,d\vec{n}, \quad \vec{F}_r = \int\int \vec{n}I\,dv\,d\vec{n}, \quad \mathbf{P}_r = \int\int \vec{n}\otimes\vec{n}I\,dv\,d\vec{n}. \tag{52}$$

The non-dimensional entropy of the radiation is obtained by computing the ratio of (22) over $\mathscr{P}\rho_\infty a_\infty^2/T_\infty$. Thus, we get the non-dimensional entropy of the radiation

$$S_r = -\frac{15}{4\pi^5}\int\int v^2(n\log n - (n+1)\log(n+1))\,dv\,d\vec{n}. \tag{53}$$

The non-dimensional entropy flux of the radiation is

$$\vec{Q}_{\mathrm{r}} = -\frac{15}{4\pi^5}\int\int v^2(n\log n - (n+1)\log(n+1))\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n}. \tag{54}$$

Here, n is already a non-dimensional variable that satisfies

$$n = \frac{4\pi^5}{15}\frac{I}{v^3}. \tag{55}$$

The non-dimensional entropy equation for smooth solutions of (44) or (51) with a zero right-hand side is

$$\frac{\partial}{\partial t}(\rho S + \mathscr{P}S_{\mathrm{r}}) + \nabla.(\rho S\vec{v} + \mathscr{P}\mathscr{C}\vec{Q}_{\mathrm{r}}) = 0. \tag{56}$$

In the sequel we shall study mainly two different regimes. The first regime is the *equilibrium diffusion regime*

$$\mathscr{P} = O(1), \quad \mathscr{C} = O(\varepsilon^{-1}), \quad \mathscr{L}_{\mathrm{s}} = O(\varepsilon^2), \quad \mathscr{L} = O(\varepsilon^{-1}),$$

the second regime is the *non-equilibrium diffusion regime*

$$\mathscr{P} = O(1), \quad \mathscr{C} = O(\varepsilon^{-1}), \quad \mathscr{L}_{\mathrm{s}} = O(\varepsilon^{-2}), \quad \mathscr{L} = O(\varepsilon^{1}).$$

We also define some moment models using the variable Eddington factor approach.

3.1. Equilibrium diffusion

In this section we perform a formal Chapman–Enskog expansion for the non-dimensional system (44).

Lemma 4. *Assume* $\mathscr{P}=1$, $\mathscr{C}=\varepsilon^{-1}$, $\mathscr{L}_{\mathrm{s}}=\varepsilon^2$, $\mathscr{L}=\varepsilon^{-1}$ *and assume the gray hypothesis* (40). *Then a first-order approximation of system* (51) *is*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + (p + p_{\mathrm{r}})\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho E + E_{\mathrm{r}}) + \nabla.((\rho E + E_{\mathrm{r}})\vec{v} + (p + p_{\mathrm{r}})\vec{v}) = \nabla.\left(\frac{1}{3\sigma_{\mathrm{a}}}\nabla T^4\right),
\end{aligned} \tag{57}$$

where

$$E_{\mathrm{r}} = T^4, \quad p_{\mathrm{r}} = \tfrac{1}{3}T^4.$$

This system is often referred to as the equilibrium-diffusion limit [3,6].

The Chapman–Enskog expansion may be split in three steps
step 1: Let us first begin with the non-dimensional transfer equation rewritten as

$$\partial_t I + \frac{1}{\varepsilon}\vec{n}.\nabla I = \frac{1}{\varepsilon^2}\frac{v_0}{v}\sigma_{\mathrm{a}}\left[\left(\frac{v}{v_0}\right)^3 B(v_0,T) - I\right] + O(1). \tag{58}$$

Here, only the absorption–emission contribution is taken into account, the scattering contribution is $O(1)$. Since we desire to equate all $O(\varepsilon^{-2})$ and $O(\varepsilon^{-1})$ terms in the equation, we have to expand the right-hand side. So (58) is rewritten as

$$\partial_t I + \frac{1}{\varepsilon}\vec{n}.\nabla I = \frac{1}{\varepsilon^2}A_{-2} + \frac{1}{\varepsilon}A_{-1} + O(1). \tag{59}$$

With straightforward notations, one has

$$\begin{aligned}
&T = T_0 + \varepsilon T_1 + O(\varepsilon^2),\\
&v = v_0 + v_0\varepsilon\vec{n}.\vec{v} + O(\varepsilon^2) \quad \text{(due to (49))},\\
&B(v_0,T) = B(v,T) - \varepsilon v\vec{n}.\vec{v}\frac{\partial}{\partial v}B(v,T) + O(\varepsilon^2),\\
&B(v,T) = B(v,T_0) + \varepsilon\frac{\partial}{\partial T}B(v,T_0)T_1 + O(\varepsilon^2),\\
&I(v,\vec{n}) = I^0 + \varepsilon I^1(v,\vec{n}) + O(\varepsilon^2).
\end{aligned} \tag{60}$$

Be careful that the intensity in the comobile frame I_0 has little to do with first-order term in the expansion of the intensity in the lab frame I_0. Expansion of the right-hand side (58) and (59) gives $A_{-2} = B(v,T_0) - I^0(v,\vec{n})$ and

$$\begin{aligned}
A_{-1} = &-\vec{n}.\vec{v}(B(v,T_0) - I^0(v,\vec{n}))\\
&+\left(\frac{\partial}{\partial T}B(v,T_0)T_1 + \vec{n}.\vec{v}(3 - v\partial_v)B(v,T_0) - I^1(v,\vec{n})\right).
\end{aligned}$$

Equating all negative powers of ε in (59), one gets

$$\begin{aligned}
&0 = B(v,T_0) - I^0(v,\vec{n}),\\
&\vec{n}.\nabla I^0 = \sigma_a\left[\frac{\partial}{\partial T}B(v,T_0)T_1 + \vec{n}.\vec{v}(3 - v\partial_v)B(v,T_0) - I^1(v,\vec{n})\right].
\end{aligned} \tag{61}$$

step 2: Next stage is to use these relations in order to expand E_r, $\vec{F}_r$ and $\mathbf{P}_r$ given by (52). For the radiation energy one obtains

$$\begin{aligned}
E_r &= \int\int I^0(v,\vec{n})\,dv\,d\vec{n} + O(\varepsilon) = \int\int B(v,T_0)\,dv\,d\vec{n} + O(\varepsilon)\\
&= \frac{15}{4\pi^5}\int\int \frac{v^3}{e^{v/T}-1}\,dv\,d\vec{n} + O(\varepsilon) = T_0^4 + O(\varepsilon).
\end{aligned} \tag{62}$$

since $\int\int [v^3/(e^v-1)]\,dv\,d\vec{n} = 4\pi^5/15$. It remains to study the radiation flux using (61)

$$\begin{aligned}
\vec{F}_r &= \int\int (I^0(v,\vec{n}) + \varepsilon I^1(v,\vec{n}))\vec{n}\,dv\,d\vec{n} + O(\varepsilon^2)\\
&= \int\int B(v,T_0)\vec{n}\,dv\,d\vec{n} + \varepsilon\int\int\left(\frac{\partial}{\partial T}B(v,T_0)T_1 + \vec{n}.\vec{v}(3 - v\partial_v)B(v,T_0) - \frac{1}{\sigma_a}\vec{n}.\nabla I^0\right)\\
&\quad\times\vec{n}\,dv\,d\vec{n} + O(\varepsilon^2).
\end{aligned} \tag{63}$$

In this expression integrals of functions linear with respect to $\vec{n}$ disappear. An elementary integration by parts gives $\int v\partial_v B(v,T_0)\,\mathrm{d}v = -\int B(v,T_0)\,\mathrm{d}v$. Since $\int\int \vec{n}\otimes\vec{n}\,\mathrm{d}v\,\mathrm{d}n = \frac{1}{3}I_d$ where I_d is the identity matrix then

$$\vec{F}_{\mathrm{r}} = \varepsilon\left[\vec{v}\frac{4}{3}T_0^4 - \frac{1}{3\sigma_{\mathrm{a}}}\nabla T_0^4\right] + O(\varepsilon^2). \tag{64}$$

Finally,

$$\mathbf{P}_{\mathrm{r}} = \int\int (I^0(v,\vec{n}) + \varepsilon I^1(v,\vec{n}))\vec{n}\otimes\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n} + O(\varepsilon^2) = \tfrac{1}{3}T_0^4 + O(\varepsilon). \tag{65}$$

step 3: We expand the first three equations of (51) using (62)–(65). Thus, one obtains the system, exact to $O(\varepsilon)$,

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho^0) + \nabla.(\rho^0\vec{v}^0) = 0,\\
&\frac{\partial}{\partial t}(\rho^0\vec{v}^0) + \nabla.(\rho^0\vec{v}^0\otimes\vec{v}^0 + (p_0 + \tfrac{1}{3}T_0^4)\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho^0E^0 + T_0^4) + \nabla.\left((\rho^0E^0 + T_0^4)\vec{v}^0 + (p_0 + \tfrac{1}{3}T_0^4)\vec{v}^0 - \frac{1}{3\sigma_{\mathrm{a}}}\nabla T_0^4\right) = 0.
\end{aligned} \tag{66}$$

The proof now ends.

3.2. Non-equilibrium diffusion

In this section, we study the non-equilibrium diffusion limit of the model. One has

Lemma 5. *Assume that* $\mathscr{P} = 1$, $\mathscr{C} = \varepsilon^{-1}$, $\mathscr{L}_{\mathrm{s}} = \varepsilon^{-2}$, $\mathscr{L} = \varepsilon^1$ *and assume the gray hypothesis* (40). *Then a first-order approximation of system* (51) *is*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + (p + p_{\mathrm{r}})\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho E + E_{\mathrm{r}}) + \nabla.((\rho E + E_{\mathrm{r}})\vec{v} + (p + p_{\mathrm{r}})\vec{v}) = \nabla.\left(\frac{1}{3\sigma_{\mathrm{s}}}\nabla T_{\mathrm{r}}^4\right),\\
&\frac{\partial}{\partial t}E_{\mathrm{r}} + \nabla.(\vec{v}E_{\mathrm{r}}) + p_{\mathrm{r}}\nabla.\vec{v} = \nabla.\left(\frac{1}{3\sigma_{\mathrm{s}}}\nabla T_{\mathrm{r}}^4\right) + \sigma_{\mathrm{a}}(T^4 - T_{\mathrm{r}}^4),
\end{aligned} \tag{67}$$

where

$$E_{\mathrm{r}} = T_{\mathrm{r}}^4, \quad p_{\mathrm{r}} = \tfrac{1}{3}T_{\mathrm{r}}^4.$$

This system is often referred to as the non-equilibrium-diffusion limit [3,6].

The scaling means that the scattering is the dominant contribution in the source term. Note that (67) has been already derived by various authors, but in Lagrangian coordinates and using more

physical intuition together with a deep understanding of the various scales of the problem than a rigorous mathematical approach. As for us we use a direct Chapman–Enskog expansion, divided in three steps. The key point is the use of formula (19) which allows to simplify the algebraic complexity of the expansion.

step 1: Following the method used in the previous section, we begin with the transfer equation in (44) rewritten as

$$\partial_t I + \frac{1}{\varepsilon}\vec{n}.\nabla I = \frac{v_0}{v}\sigma_a(v_0)\left[\left(\frac{v}{v_0}\right)^3 B(v_0,T) - I\right] + \frac{1}{\varepsilon^2}\frac{v_0}{v}\sigma_s(v_0)\left[\left(\frac{v}{v_0}\right)^3 \frac{1}{4\pi}\int \frac{v_0}{v'} I(v',\vec{n}')\,\mathrm{d}\vec{n}' - I\right]. \tag{68}$$

We want to rewrite this expression as

$$\partial_t I + \frac{1}{\varepsilon}\vec{n}.\nabla I = \frac{1}{\varepsilon^2}A_{-2} + \frac{1}{\varepsilon}A_{-1} + A_0 + O(\varepsilon). \tag{69}$$

So we use (60) and expand the scattering

$$\left[\left(\frac{v}{v_0}\right)^3 \frac{1}{4\pi}\int \frac{v_0}{v'} I(v',\vec{n}')\,\mathrm{d}\vec{n}' - I\right] = C_{-2} + \varepsilon C_{-1} + \varepsilon^2 C_0 + O(\varepsilon^3). \tag{70}$$

Then

$$A_{-2} = \sigma_s C_{-2} = \sigma_s\left(\frac{1}{4\pi}\int I^0(v,\vec{n})\,\mathrm{d}\vec{n} - I^0\right),$$

$$A_{-1} = \sigma_s C_{-1} = \sigma_s\left(-\frac{1}{4\pi}\int \vec{n}.\vec{v}I^0(v,\vec{n})\,\mathrm{d}\vec{n} + 3\vec{n}.\vec{v}\frac{1}{4\pi}\int I^0(v,\vec{n})\,\mathrm{d}\vec{n} + \frac{1}{4\pi}\int v(\vec{n}'.\vec{v} - \vec{n}.\vec{v})\frac{\partial I^0(v,\vec{n}')}{\partial v}\,\mathrm{d}\vec{n}' + \frac{1}{4\pi}\int I^1(v,\vec{n})\,\mathrm{d}\vec{n} - I^1 - \vec{v}.\vec{n}C_{-2}\right),$$

and

$$A_0 = \sigma_s C_0 + \sigma_a(B(v,T) - I^0). \tag{71}$$

The contribution $\int v(\vec{n}'.\vec{v} - \vec{n}.\vec{v})(\partial I^0(v,\vec{n}')/\partial v)\,\mathrm{d}\vec{n}'$ is the consequence of

$$I(v',\vec{n}') = I\left(v\frac{1-\varepsilon\vec{n}.\vec{v}}{1-\varepsilon\vec{n}'.\vec{v}},\vec{n}'\right) = I(v,\vec{n}') + \varepsilon v(\vec{n}'.\vec{v} - \vec{n}.\vec{v})\frac{\partial I(v,\vec{n}')}{\partial v} + O(\varepsilon^2).$$

Now equating the ε^{-2} terms in Eq. (68), we get

$$C_{-2} = \frac{1}{4\pi}\int I^0(v,\vec{n})\,\mathrm{d}\vec{n} - I^0 = 0,$$

which means that the radiation is isotropic $I^0(v,\vec{n}) = I^0(v)$. Equating the ε^{-1} terms in Eq. (68) and using $\int I^0(v,\vec{n})\vec{n}\,\mathrm{d}\vec{n} = 0$ plus $\int v(\vec{n}'.\vec{v} - \vec{n}.\vec{v})(\partial I^0(v,\vec{n}')/\partial v)\,\mathrm{d}\vec{n}' = -\vec{n}.\vec{v}\int v(\partial I^0(v,\vec{n})/\partial v)\,\mathrm{d}\vec{n}$, we get

$$\sigma_s C_{-1} = \vec{n}.\nabla I^0 = \sigma_s \left(3\vec{n}.\vec{v}\,\frac{1}{4\pi}\int I^0(v,\vec{n})\,\mathrm{d}\vec{n} - \frac{1}{4\pi}\,\vec{n}.\vec{v}\int v\,\frac{\partial I^0(v,\vec{n})}{\partial v}\,\mathrm{d}\vec{n} + \frac{1}{4\pi}\int I^1(v,\vec{n})\,\mathrm{d}\vec{n} - I^1\right). \tag{72}$$

It remains to express the ε^0 terms in expression (68). In order to simplify the analysis which can be very cumbersome, we use the property (19) which expresses the fact that the scattering contribution is isotropic in the comobile reference frame. So

$$\int\int \frac{v_o}{v}\,(C_{-2} + \varepsilon C_{-1} + \varepsilon^2 C_0 + O(\varepsilon^3))\,\mathrm{d}v\,\mathrm{d}\vec{n} = 0.$$

Since $v_0/v = (1 - \varepsilon\vec{n}.\vec{v})/\gamma$, then we are able to expand in power of ε. We get three formulas. The first one is $\int\int C_{-2}\,\mathrm{d}v\,\mathrm{d}\vec{n} = 0$, the second one is $\int\int C_{-1}\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int\int \vec{n}.\vec{v}C_{-2}\,\mathrm{d}v\,\mathrm{d}\vec{n}$ and the third one is

$$\int\int C_0\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int\int \vec{n}.\vec{v}C_{-1}\,\mathrm{d}v\,\mathrm{d}\vec{n}. \tag{73}$$

Combining (73) with (72), it gives

$$\sigma_s \int\int C_0\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int\int (\vec{n}.\vec{v})(\vec{n}.\nabla I^0)\,\mathrm{d}v\,\mathrm{d}\vec{n} = \tfrac{1}{3}\vec{v}.\int \nabla I^0\,\mathrm{d}v. \tag{74}$$

Here we have used the isotropy of I_0 and

$$\int\int \vec{n}\otimes\vec{n}\,\mathrm{d}\vec{n} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{3}\,\mathbf{I}. \tag{75}$$

This expression will be used to simplify the integrated form of (71).

step 2: Now we use all this informations in order to calculate E_r and F_r.

First, we define $T_r^4 = \int\int I^0(v,\vec{n})\,\mathrm{d}v\,\mathrm{d}\vec{n}$, so

$$E_r = \int\int I^0(v,\vec{n})\,\mathrm{d}v\,\mathrm{d}\vec{n} + O(\varepsilon) = T_r^4 + O(\varepsilon). \tag{76}$$

Eq. (74) can be also written as

$$\sigma_s \int\int C_0\,\mathrm{d}v\,\mathrm{d}\vec{n} = \vec{v}.\nabla p_r, \tag{77}$$

where by definition $p_r = \frac{1}{3}T_r^4$. Then we compute $\vec{F}_r$

$$\vec{F}_r = \int\int (I^0(v,\vec{n}) + \varepsilon I^1(v,\vec{n}))\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n} + O(\varepsilon^2),$$

$$\vec{F}_r = \varepsilon\int\int \left[\vec{n}.\vec{v}\,\frac{1}{4\pi}\int\left(3 - v\frac{\partial}{\partial v}\right)I^0(v,\vec{n})\,\mathrm{d}\vec{n} + \frac{1}{4\pi}\int I^1(v,\vec{n})\,\mathrm{d}\vec{n} - \frac{1}{\sigma_s}\,\vec{n}.\nabla I^0\right]\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n} + O(\varepsilon^2).$$

Using an integration by part to get rid of the $\nu\partial/\partial\nu$ we get

$$\vec{F}_{\rm r} = \varepsilon\left[\vec{v}\,\frac{4}{3}\,T_{\rm r}^4 - \frac{1}{3\sigma_{\rm s}}\,\nabla T_{\rm r}^4\right] + O(\varepsilon^2). \tag{78}$$

This formula is the same as (64), except that the temperature of the fluid T_0 is replaced by what will be referred to as the temperature of the radiation $T_{\rm r}$. Similarly to (65) one has

$$\mathbf{P}_{\rm r} = \int\int (I^0(\nu,\vec{n}) + \varepsilon I^1(\nu,\vec{n}))\vec{n}\otimes\vec{n}\,{\rm d}\nu\,{\rm d}\vec{n} + O(\varepsilon^2) = \tfrac{1}{3}\,T_{\rm r}^4 + O(\varepsilon). \tag{79}$$

step 3: It remains to expand (44) using (76)–(79). One gets exactly the same result as in the equilibrium regime, except that T_0 is replace by $T_{\rm r}$. Thus, one obtains the system, exact to $O(\varepsilon)$,

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho^0) + \nabla.(\rho^0\vec{v}^0) = 0,\\
&\frac{\partial}{\partial t}(\rho^0\vec{v}^0) + \nabla.(\rho^0\vec{v}^0\otimes\vec{v}^0 + (p_0 + \tfrac{1}{3}T_{\rm r}^4)\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho^0E^0 + T_{\rm r}^4) + \nabla.\left((\rho^0E^0 + T_{\rm r}^4)\vec{v}^0 + (p_0 + \tfrac{1}{3}\,T_{\rm r}^4)\vec{v}^0 - \frac{1}{3\sigma_{\rm s}}\nabla T_{\rm r}^4\right) = 0.
\end{aligned} \tag{80}$$

Finally, we need an equation for $T_{\rm r}$ in order to close (80). For this we integrate the equation of transfer and consider only the $O(1)$ contribution. It gives

$$\partial_t\int\int I^0\,{\rm d}\nu\,{\rm d}\vec{n} + \nabla\int\int I\vec{n}\,{\rm d}\nu\,{\rm d}\vec{n} = \int\int A_0\,{\rm d}\nu\,{\rm d}\vec{n} + O(\varepsilon). \tag{81}$$

Since A_0 is given by (71), one has

$$\begin{aligned}
\int\int A_0\,{\rm d}\nu\,{\rm d}\vec{n} &= \sigma_{\rm a}\int\int (B(\nu,T) - I_0)\,{\rm d}\nu\,{\rm d}\vec{n} + \sigma_{\rm s}\int\int C_0\,{\rm d}\nu\,{\rm d}\vec{n} + O(\varepsilon)\\
&= \sigma_{\rm a}(T_0^4 - T_{\rm r}^4) + \vec{v}.\nabla p_{\rm r} + O(\varepsilon).
\end{aligned}$$

Using the value of $\nabla\int\int I^1\vec{n}\,{\rm d}\nu\,{\rm d}\vec{n}$ given in (78), one gets

$$\partial_t E_{\rm r} + \nabla.\left(\vec{v}\,\frac{4}{3}\,T_{\rm r}^4 - \frac{1}{3\sigma_{\rm s}}\,\nabla T_{\rm r}^4\right) = \sigma_{\rm a}(T_0^4 - T_{\rm r}^4) + \vec{v}.\nabla p_{\rm r} + O(\varepsilon)$$

finally rewritten as

$$\partial_t E_{\rm r} + \nabla.(\vec{v}E_{\rm r}) + p_{\rm r}\nabla.\vec{v} = \sigma_{\rm a}(T_0^4 - T_{\rm r}^4) + \nabla.\left(\frac{1}{3\sigma_{\rm s}}\,\nabla T_{\rm r}^4\right) + O(\varepsilon). \tag{82}$$

In conjunction with (80), we have the proof.

Corollary 1. *If one uses a non-relativistic source term in the right-hand side of the transfer equation* (51), *then one misses the* $p_{\rm r}\nabla.\vec{v}$ *in the non-equilibrium diffusion model* (67).

Non-relativistic means that one equates $\nu = \nu_0$ in the definition of the source term, even for $\vec{v} \neq 0$. So a non-relativistic scattering source term will be isotropic in the lab frame, and not in the comobile

frame as in (19). Since $p_r\nabla.\vec{v}$ is a direct consequence of formula (19), the proof of the corollary ends.

Corollary 2. *Consider the non-equilibrium diffusion model* (67). *Let us define*

$$\bar{S}_r = \tfrac{4}{3} T_r^3. \tag{83}$$

Then (67) *may be rewritten as*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + (p+p_r)\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho E + E_r) + \nabla.((\rho E + E_r)\vec{v} + (p+p_r)\vec{v} = \nabla.\left(\frac{1}{3\sigma_s}\nabla T_r^4\right),\\
&\frac{\partial}{\partial t}\bar{S}_r + \nabla.(\vec{v}\bar{S}_r) = \frac{1}{T_r}\nabla.\left(\frac{1}{3\sigma_s}\nabla T_r^4\right) + \sigma_a\frac{T^4 - T_r^4}{T_r}.
\end{aligned} \tag{84}$$

The quantity $\bar{S}_r$ is formerly the radiative entropy at equilibrium. But since S_r has been already defined in (22) then $\bar{S}_r$ is for the moment different from S_r. We will see in Lemma 7 that these quantities are in some sense equal. The proof of corollary 2 is just a matter of direct calculation. We just divide the last equation of (67) by T_r and rearrange all terms due to

$$\begin{aligned}
\frac{1}{T_r}\left(\frac{\partial}{\partial t}E_r + \nabla.(\vec{v}E_r) + p_r\nabla\vec{v}\right) &= \frac{1}{T_r}\left(\frac{\partial}{\partial t}T_r^4 + \nabla T_r^4.\vec{v} + T_r^4\nabla.\vec{v} + \frac{1}{3}T_r^4\nabla.\vec{v}\right)\\
&= \frac{\partial}{\partial t}\left(\frac{4}{3}T_r^3\right) + \nabla\left(\frac{4}{3}T_r^3\right).\vec{v} + \frac{4}{3}T_r^3\nabla.\vec{v} = \frac{\partial}{\partial t}\bar{S}_r + \nabla.(\vec{v}\bar{S}_r).
\end{aligned}$$

An important advantage of (84) against (67) is that it admits a natural conservative limit for

$$\sigma_s \approx +\infty \quad \text{and} \quad \sigma_a \approx 0. \tag{85}$$

Indeed the limit is the system of conservation laws

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}(\rho\vec{v}) + \nabla.(\rho\vec{v}\otimes\vec{v} + (p+p_r)\mathbf{I}) = 0,\\
&\frac{\partial}{\partial t}(\rho E + E_r) + \nabla.((\rho E + E_r)\vec{v} + (p+p_r)\vec{v}) = 0,\\
&\frac{\partial}{\partial t}\bar{S}_r + \nabla.(\vec{v}\bar{S}_r) = 0.
\end{aligned} \tag{86}$$

This system admits discontinuous solutions (*à la* Rankine–Hugoniot), which is not the case for (67) due to the non-conservative term $p_r\nabla.\vec{v}$. This remark will be used in our last section.

3.3. P^1 moments models

Let us now recall briefly another type of approximate transport models, the P^1 moment models. By taking the moments of the radiative transfer equation in (44) against 1 and $\vec{n}$, one obtain the general form of P^1 moment models

$$\frac{1}{\mathscr{C}}\frac{\partial}{\partial t}E_{\mathrm{r}}+\nabla\vec{F}_{\mathrm{r}}=S_E, \tag{87}$$

$$\frac{1}{\mathscr{C}}\frac{\partial}{\partial t}\vec{F}_{\mathrm{r}}+\nabla\mathbf{P}_{\mathrm{r}}=\vec{S}_F. \tag{88}$$

It is expected that such moment models should give a good approximation of the solution of the radiative transfer equation in (44) in the case of small anisotropy (that is $\|\vec{F}_{\mathrm{r}}/E_{\mathrm{r}}\|$ small). But they are widely used in the full range of anisotropy (that is for $\|\vec{F}_{\mathrm{r}}/E_{\mathrm{r}}\|\leqslant 1$). However, one must close the system (87) and (88). This is done with a formula where $\mathbf{P}_{\mathrm{r}}$ is given in terms of E_{r} and $\vec{F}_{\mathrm{r}}$.

A popular closure is called the *Variable Eddington Factor*. In this method one takes $\mathbf{P}_{\mathrm{r}}$ as

$$\mathbf{P}_{\mathrm{r}}=E_{\mathrm{r}}D_{\mathrm{r}}=E_{\mathrm{r}}\left(\frac{1-\chi}{2}\mathbf{I}+\frac{3\chi-1}{2}f\otimes f\right), \tag{89}$$

where $\vec{f}=\vec{F}_{\mathrm{r}}/E_{\mathrm{r}}$: $\chi=\chi(\|f\|)$ is the *Eddington factor* and is usually chosen as a function of $\|\vec{f}\|$. Then the job consist to specify the Eddington factor. As we can notice in the literature, there is a lot of propositions. Considering the solution of the transfer equation, it seems natural to choose an Eddington factor which satisfies the constraints $\|\vec{f}\|^2\leqslant\chi(\|\vec{f}\|)\leqslant 1$, $\chi(0)=1/3$, $\chi(1)=1$ (we refer to [14]).

The Eddington approximation, which correspond to a constant Eddington factor $\chi=\frac{1}{3}$ does not fulfill all these requirements: it leads to solutions for which $\|f\|>1$, see [15]. We refer to [16] or [14] for different types of closure. To derive a P^1 approximation of the relativistic transfer equation in (44), we will use a method based on maximizing the entropy under constraints and which was already used in the non-relativistic case by several authors [13,14,17,18]. As we will see in the next, this approach gives a well-known Eddington factor, see [7,14,18].

$$\chi(x)=\frac{3+4x^2}{5+2\sqrt{4-3x^2}}. \tag{90}$$

Another difficulty with the variable Eddington factor approach is that one must deal with the relativistic source terms and approximate these terms by simpler ones, essentially by expanding them using Taylor expansion in power of $|v|/c$. All the discussion is about which term must be kept to give the good equilibrium state.

As for us we consider that such approximate models must contain the equilibrium diffusion approximation. Our contribution in this paper is to show how to design P^1 approximate models which permit also to recover the non-equilibrium diffusion approximation. To our understanding this is not the case for some previous works, see for example [3,13]. For example, the P^1 model used by Feugeas and Dubroca [13] coupled with the Euler equations, for the matter, cannot give the $p_{\mathrm{r}}\nabla.\vec{v}$ term in the non-equilibrium diffusion approximation since the source term have been treated in a non-relativistic way (it is an application of the Corollary 1).

4. Method of moments

The method of moments applied to systems of conservation laws (we refer to [14,19] in a particular context [15], and references therein) amounts to the derivation of reduced and well-posed systems of conservations laws: reduced means that the number of equations of the reduced system is smaller than for the original system; well posed means that the reduced system is still hyperbolic if the original system was hyperbolic. Applications of the method of moments to the transfer equation may be founded in [13,17].

4.1. Generality about the method of moments for radiation hydrodynamics

Let us consider the non-dimensional radiative entropy S_r defined by (53).

Lemma 6. *Let* $(E_\mathrm{r}, \vec{F}_\mathrm{r}) \in \mathbb{R} \times \mathbb{R}^3$ *with* $E_\mathrm{r} > 0$ *and* $|\vec{F}_\mathrm{r}| \leqslant E_\mathrm{r}$. *The minimum of the radiative entropy* S_r *with constraints*

$$\int\int I \,\mathrm{d}v\,\mathrm{d}\vec{n} = E_\mathrm{r} \quad \textit{and} \quad \int\int I\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n} = \vec{F}_\mathrm{r} \tag{91}$$

is given by

$$n = \frac{1}{e^{(v/\Theta_\mathrm{r}) + v\vec{b}.\vec{n}/\Theta_\mathrm{r}} - 1} \tag{92}$$

where $\Theta_\mathrm{r} > 0$ *and* $|\vec{b}| < 1$.

Then the pressure is given by the well-known formula (*Levermore* [14], *see also in the appendix*)

$$\mathbf{P}_\mathrm{r} = E_\mathrm{r} D_\mathrm{r} = E_\mathrm{r}\left(\frac{1-\chi}{2}\mathbf{I} + \frac{3\chi - 1}{2}\vec{f} \otimes \vec{f}\right) \tag{93}$$

with $\vec{f} = \vec{F}_\mathrm{r}/E_\mathrm{r}$ *and the Eddington factor* χ *given by*

$$\chi = \frac{3 + 4||\vec{f}||^2}{5 + 2\sqrt{4 - 3||\vec{f}||^2}}. \tag{94}$$

Since S_r is strictly concave with respect to I, it is sufficient to check the optimality conditions, see [13,17] for more details. We construct the Lagrangian with Lagrange multipliers $1/\Theta_\mathrm{r} \in \mathbb{R}$ and $\vec{b}/\Theta_\mathrm{r} \in \mathbb{R}^3$

$$L = S_R - \frac{1}{\Theta_\mathrm{r}} E_\mathrm{r} - \frac{\vec{b}}{\Theta_\mathrm{r}}.\vec{F}_\mathrm{r}.$$

The optimality conditions are simply

$$0 = \mathrm{d}L = \mathrm{d}S_\mathrm{r} - \frac{1}{\Theta_\mathrm{r}}\mathrm{d}E_\mathrm{r} - \frac{\vec{b}}{\Theta_\mathrm{r}}\mathrm{d}\vec{F}_\mathrm{r}. \tag{95}$$

Due to (52) and $dS_r = \int_\nu \int_{\vec{n}} (1/\nu) \log(n/(n+1))\, dI\, d\nu\, d\vec{n}$, one gets

$$0 = -\int_\nu \int_{\vec{n}} \left[\frac{1}{\nu} \log\left(\frac{n}{n+1}\right) + \frac{1}{\Theta_r} + \frac{\vec{b}}{\Theta_r}.\vec{n} \right] dI\, d\nu\, d\vec{n}.$$

Thus, one has $\log(n/(n+1)) + \nu/\Theta_r + (\nu\vec{b}/\Theta_r).\vec{n} = 0$, that is $n/(n+1) = e^{-(\nu/\Theta_r + \nu\vec{b}.\vec{n}/\Theta_r)}$ which is equivalent to (92). Since $n \geqslant 0$ for all $(\nu, \vec{n})$ then we need the compatibility condition $\Theta_r > 0$ and $|\vec{b}| < 1$. The proof now ends.

Then one defines the reduced and closed system

$$\begin{aligned} &\frac{1}{\mathscr{C}} \frac{\partial}{\partial t} E_r + \nabla \vec{F}_r = S_E, \\ &\frac{1}{\mathscr{C}} \frac{\partial}{\partial t} \vec{F}_r + \nabla \mathbf{P}_r = \vec{S}_F. \end{aligned} \tag{96}$$

These four equations (one for E_r and three for $\vec{F}_r$) are the moments of (17) against $(1, \vec{n}_1, \vec{n}_2, \vec{n}_3) = (1, \vec{n})$. System (96) is closed in the sense that (96) contains four equations while the intensity is given by four degrees of freedom (see (98)). The complete moment model deduced from (96) and (51) is then

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0, \\ &\frac{\partial}{\partial t}\left(\rho\vec{v} + \frac{\mathscr{P}}{\mathscr{C}} \vec{F}_r\right) + \nabla.(\rho\vec{v} \otimes \vec{v} + p\mathbf{I} + \mathscr{P}\mathbf{P}_r) = 0, \\ &\frac{\partial}{\partial t}(\rho E + \mathscr{P} E_r) + \nabla.(\rho E\vec{v} + p\vec{v} + \mathscr{P}\mathscr{C}\vec{F}_r) = 0, \\ &\frac{1}{\mathscr{C}} \frac{\partial}{\partial t} E_r + \nabla \vec{F}_r = S_E, \\ &\frac{1}{\mathscr{C}} \frac{\partial}{\partial t} \vec{F}_r + \nabla.\mathbf{P}_r = \vec{S}_F, \end{aligned} \tag{97}$$

together with

$$\frac{4\pi^5}{15} \frac{I}{\nu^3} = \frac{1}{e^{(\nu/\Theta_r) + \nu\vec{b}.\vec{n}/\Theta_r} - 1}. \tag{98}$$

4.2. Chapman–Enskog expansion of the moment model

In this section we prove that the moment model (97) contains the non-equilibrium diffusion in the sense that (67) is recovered as the Chapman–Enskog expansion of the moment model.

Lemma 7. *Assume that $\mathscr{P} = 1$, $\mathscr{C} = \varepsilon^{-1}$, $\mathscr{L}_s = \varepsilon^{-2}$, $\mathscr{L} = \varepsilon^1$ and assume the gray hypothesis (40). Then a first-order approximation of system (97) is the non-equilibrium diffusion model (67). The*

coefficients of the radiation (see (98)) are

$$\Theta_r = T_r + O(\varepsilon), \quad \vec{b} = O(\varepsilon). \tag{99}$$

The second equation $\vec{b} = O(\varepsilon)$ *means that the radiation is isotropic at the limit. We also get* $S_r = \bar{S}_r + O(\varepsilon)$ *where* $\bar{S}_r = \frac{4}{3} T_r^3$ (83).

This result means that we have not lost too much informations by taking the first two moments of the transfer equation. At least we are able to recover the non-equilibrium diffusion limit. It also justifies the use of $\bar{S}_r$, which is the radiative entropy, in (84). One must be convinced that this result is not trivial. Indeed, one never assumes in the analysis of the non-equilibrium diffusion limit that radiation is closed to a Planckian with a radiative temperature T_r. So the exact shape of the radiation I is not addressed in the non-equilibrium diffusion limit, even if the model behaves just as if the radiation intensity is closed to a Planckian around T_r. On the other hand, the moment model assumes such a representation for the intensity, which is a generalized Planckian. So one might loose too much information with the moment model compared with the non-equilibrium diffusion model. The lemma shows it is not the case. The proof essentially uses the same method as in the proof of Lemma 5.

Step 1: Of course we expand

$$E_r = E_{r0} + \varepsilon E_{r1} + O(\varepsilon^2), \quad \vec{F}_r = \mathbf{F}_{r0} + \varepsilon \mathbf{F}_{r1} + O(\varepsilon^2)$$

and

$$\Theta_r = \Theta_{r0} + O(\varepsilon), \quad \vec{b} = \vec{b}_0 + O(\varepsilon).$$

Thus, the intensity of radiation is $I = I^0 + \varepsilon I^1 + o(\varepsilon)$ where

$$I^0 = \frac{15}{4\pi^5} v^3 \frac{1}{e^{v/\Theta_{r0} + v\vec{b}_0.\vec{n}/\Theta_{r0}} - 1}. \tag{100}$$

Comparing the system (97) and (69) and using the same scaling, we deduce some equalities

$$\int\int C_{-2}\, dv\, d\vec{n} = 0, \quad \int\int C_{-2}\vec{n}\, dv\, d\vec{n} = 0 \quad (\text{order } \varepsilon^{-2}), \tag{101}$$

$$\int\int \vec{n}.\nabla I_0\, dv\, d\vec{n} = \sigma_s \int\int C_{-1}\, dv\, d\vec{n} = 0 \tag{102}$$

and

$$\int\int \vec{n} \otimes \vec{n} \nabla I_0\, dv\, d\vec{n} = \sigma_s \int\int C_{-1}\vec{n}\, dv\, d\vec{n} = 0 \quad (\text{order } \varepsilon^{-1}), \tag{103}$$

where C_{-2} and C_{-1} are defined in (69). The first equation of (101) is of course always true. The second equation of (101) implies that

$$-\int\int \vec{n} I^0\, dn = \int\int \vec{n}\left(\frac{1}{4\pi}\int I^0(v,\vec{n})\, d\vec{n} - I^0\right) = 0.$$

It is equivalent (see the appendix) to

$$0 = \frac{4}{3 + |\vec{b}_0|^2}\left(\int\int I^0 \,\mathrm{d}\vec{n}\right)\vec{b}_0$$

which in turn implies $\vec{b}_0 = 0$: the first-order term of the radiation is isotropic. Eqs. (102) and (103) are the integrated counterpart of (72). Since (73) is still true and the first-order term of the radiation is isotropic we also deduce (74).

Steps 2 and 3 are identical to steps (2) and (3) in the proof of Lemma 5.

Step 4: We have to check (99): since $\vec{b}_0 = 0$ we already have $\vec{b} = O(\varepsilon)$. Due to (100) one has $\int\int I \,\mathrm{d}\nu\,\mathrm{d}\vec{n} = \Theta_{\mathrm{r}0}^4 + O(\varepsilon)$. Compared with the definition of T_{r} (76) we get $\Theta_{\mathrm{r}} = T_{\mathrm{r}} + O(\varepsilon)$. The proof now ends.

4.3. An alternative moment model

In this section we use an elementary algebraic relation in order to rewrite the moment model (97) using $(S_{\mathrm{r}}, \vec{F}_{\mathrm{r}})$ instead of $(E_{\mathrm{r}}, \vec{F}_{\mathrm{r}})$. Of course it is possible to use the general theory of the method of moment to prove the result but the proof presented here has the advantage to be self contained.

Lemma 8. *Smooth solutions of* (97) *are also smooth solutions of*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \nabla.(\rho\vec{v}) = 0,\\
&\frac{\partial}{\partial t}\left(\rho\vec{v} + \frac{\mathscr{P}}{\mathscr{C}}\vec{F}_{\mathrm{r}}\right) + \nabla.(\rho\vec{v}\otimes\vec{v} + p\mathbf{I} + \mathscr{P}\mathbf{P}_{\mathrm{r}}) = 0,\\
&\frac{\partial}{\partial t}(\rho E + \mathscr{P}E_{\mathrm{r}}) + \nabla.(\rho E\vec{v} + p\vec{v} + \mathscr{P}\mathscr{C}\vec{F}_{\mathrm{r}}) = 0,\\
&\frac{1}{\mathscr{C}}\frac{\partial}{\partial t}S_{\mathrm{r}} + \nabla.\vec{Q}_{\mathrm{r}} = \frac{1}{\Theta_{\mathrm{r}}}(S_E + \vec{b}.\vec{S}_F),\\
&\frac{1}{\mathscr{C}}\frac{\partial}{\partial t}\vec{F}_{\mathrm{r}} + \nabla.\mathbf{P}_{\mathrm{r}} = \vec{S}_F.
\end{aligned} \tag{104}$$

Due to (95) one has

$$\partial_t S_{\mathrm{r}} = \frac{1}{\Theta_{\mathrm{r}}}\partial_t E_{\mathrm{r}} + \frac{\vec{b}}{\Theta_{\mathrm{r}}}.\partial_t\vec{F}_{\mathrm{r}}. \tag{105}$$

It is easy to get a similar formula for the entropy flux. The non-dimensional entropy flux is $\vec{Q}_{\mathrm{r}} = -(15/4\pi^5)\int\int \nu^2(n\log n - (n+1)\log(n+1))\vec{n}\,\mathrm{d}\nu\,\mathrm{d}\vec{n}$. Thus,

$$\begin{aligned}
\mathrm{d}\vec{Q}_{\mathrm{r}} &= \int\int \frac{1}{\nu}\log\left(\frac{n}{n+1}\right)\mathrm{d}I\vec{n}\,\mathrm{d}\nu\,\mathrm{d}\vec{n}\\
&= \int\int \frac{1}{\nu}\left(\frac{1}{\Theta_{\mathrm{r}}} + \frac{\vec{b}.\vec{n}}{\Theta_{\mathrm{r}}}\right)\mathrm{d}I\vec{n}\,\mathrm{d}\nu\,\mathrm{d}\vec{n} = \frac{1}{\Theta_{\mathrm{r}}}\mathrm{d}\vec{F}_{\mathrm{r}} + \mathrm{d}\mathbf{P}_{\mathrm{r}}\frac{\vec{b}}{\Theta_{\mathrm{r}}}.
\end{aligned}$$

So we have for partial derivatives in space

$$\nabla.\vec{Q}_{\mathrm{r}} = \sum_{j=1}^{3} \partial_j \vec{Q}_{\mathrm{r}}^j = \sum_{j=1}^{3} \left(\frac{1}{\Theta_{\mathrm{r}}} \partial_j \vec{F}_{\mathrm{r}}^j + \sum_{k=1}^{3} \partial_j \mathbf{P}_{\mathrm{r}}^{jk} \frac{\vec{b}^k}{\Theta_{\mathrm{r}}} \right). \tag{106}$$

Since the pressure tensor $\mathbf{P}_{\mathrm{r}}$ is symmetric, then $\mathbf{P}_{\mathrm{r}}^{jk} = \mathbf{P}_{\mathrm{r}}^{kj}$. Thus, (106) is equal to

$$\nabla.\vec{Q}_{\mathrm{r}} = \frac{1}{\Theta_{\mathrm{r}}} \nabla.F_{\mathrm{r}} + \frac{\vec{b}}{\Theta_{\mathrm{r}}}.(\nabla.\mathbf{P}_{\mathrm{r}}). \tag{107}$$

Combining (97) and (105)–(107), the result of the lemma is now straightforward. In the rest of this paper (104a) is referred to as the modified moment model, as opposed to the moment model (97).

5. Rankine–Hugoniot relations

In this section we advocate that, in some regimes, (104) is probably physically more relevant than (97). We base the discussion on *discontinuous solutions* of (97) and (104). From the general theory of hyperbolic systems of conservations laws we already know that discontinuous solutions of (97) and (104) are different. But to what amount? If the difference tends to zero in the non-equilibrium limit, we have to conclude that the difference is non-essential in this non-equilibrium regime, and that both (97) and (104) can be used. The main result of this section is that the difference is large, and that the correct approximation is the modified moment model (104).

Let $\vec{d}$ be the normal derivative on a line of discontinuity of the solution and let σ be the velocity of the line of discontinuity. In the following NE (resp. MM, mMM) stands for Non-Equilibrium (86) model (resp. Moment (97) Model, modified Moment (104) Model). The Rankine–Hugoniot relations for these models are

$$(\mathrm{NE}) \begin{cases} -\sigma[\rho] + \vec{d}.[\rho\vec{v}] = 0, \\ -\sigma[\rho\vec{v}] + \vec{d}.[\rho\vec{v} \otimes \vec{v} + (p + p_{\mathrm{r}})\mathbf{I}] = 0, \\ -\sigma[\rho E + E_{\mathrm{r}}] + \vec{d}.[(\rho E + E_{\mathrm{r}})\vec{v} + (p + p_{\mathrm{r}})\vec{v}] = 0, \\ -\sigma[S_{\mathrm{r}}] + \vec{d}.[\vec{v} S_{\mathrm{r}}] = 0, \end{cases} \tag{108}$$

$$(\mathrm{MM}) \begin{cases} -\sigma[\rho] + d.[\rho\vec{v}] = 0, \\ -\sigma \left[\rho\vec{v} + \dfrac{\mathscr{P}}{\mathscr{C}} \vec{F}_{\mathrm{r}} \right] + \vec{d}.[\rho\vec{v} \otimes \vec{v} + p\mathbf{I} + \mathscr{P}\mathbf{P}_{\mathrm{r}}] = 0, \\ -\sigma[\rho E + \mathscr{P} E_{\mathrm{r}}] + \vec{d}.[\rho E\vec{v} + p\vec{v} + \mathscr{P}\mathscr{C}\vec{F}_{\mathrm{r}}] = 0, \\ -\sigma[E_{\mathrm{r}}] + \vec{d}.[\mathscr{C}\vec{F}_{\mathrm{r}}] = 0, \\ -\sigma[\vec{F}_{\mathrm{r}}] + \vec{d}.[\mathscr{C}\mathbf{P}_{\mathrm{r}}] = 0, \end{cases} \tag{109}$$

and

$$(\mathrm{mMM})\begin{cases} -\sigma[\rho]+d.[\rho\vec{v}]=0,\\ -\sigma\left[\rho\vec{v}+\dfrac{\mathscr{P}}{\mathscr{C}}\vec{F}_{\mathrm{r}}\right]+\vec{d}.[\rho\vec{v}\otimes\vec{v}+p\mathbf{I}+\mathscr{P}\mathbf{P}_{\mathrm{r}}]=0,\\ -\sigma[\rho E+\mathscr{P}E_{\mathrm{r}}]+\vec{d}.[\rho E\vec{v}+p\vec{v}+\mathscr{P}\mathscr{C}\vec{F}_{\mathrm{r}}]=0,\\ -\sigma[S_{\mathrm{r}}]+\vec{d}.[\mathscr{C}\vec{Q}_{\mathrm{r}}]=0,\\ -\sigma[\vec{F}_{\mathrm{r}}]+\vec{d}.[\mathscr{C}\mathbf{P}_{\mathrm{r}}]=0, \end{cases} \tag{110}$$

In all these expressions $[f]$ stands for the difference of the left and right state across the discontinuity line: $[f]=f_{\mathrm{R}}-f_{\mathrm{L}}$. All these systems of Rankine–Hugoniot relations must be supplemented by entropy inequalities. We observe that

Lemma 9 (Compatibility of the modified moment model with the non-equilibrium model). *Let us consider the Rankine–Hugoniot relations of the modified moment model* (110) *between a left and a right state for which the regime is* $\mathscr{P}=1$, $\mathscr{C}=\varepsilon^{-1}$, $\mathscr{L}_{\mathrm{s}}=\varepsilon^{-4}$, $\mathscr{L}=\varepsilon^{2}$: *note that this regime is compatible with the assumption* (85) *already encountered in the study of the non-equilibrium limit. We consider a shock velocity* $\sigma=O(1)$ *which means that we are interested only in shocks at moderate velocities* $O(1)$ *in the lab frame. Then*

(a) *An* $O(\varepsilon)$ *approximation of* (110) *is* (108). *The equation for the radiative temperature at discontinuities is*

$$-\sigma[T_{\mathrm{r}}^{3}]+\vec{d}.[\vec{v}T_{\mathrm{r}}^{3}]=0. \tag{111}$$

(b) *The system* (108) *is not an* $O(\varepsilon)$ *of* (109), *since the limit equation for the temperature is*

$$-\sigma[T_{\mathrm{r}}^{4}]+\vec{d}.\left[\frac{4}{3}\vec{v}T_{\mathrm{r}}^{4}\right]=0. \tag{112}$$

The method we use is of course based on three Chapman–Enskog expansions: a Chapman–Enskog expansion for the left state; another for the right state: the last one for the Rankine–Hugoniot relations. In order to simplify the discussion we remark that the regime $\mathscr{P}=1$, $\mathscr{C}=\varepsilon^{-1}$, $\mathscr{L}_{\mathrm{s}}=\varepsilon^{-4}$, $\mathscr{L}=\varepsilon^{2}$ is equivalent to $\mathscr{P}=1$, $\mathscr{C}=\varepsilon^{-1}$, $\mathscr{L}_{\mathrm{s}}=\varepsilon^{-2}$, $\mathscr{L}=\varepsilon^{1}$ plus $\sigma_{\mathrm{s}}=O(\varepsilon^{-1})$ and $\sigma_{\mathrm{a}}=O(\varepsilon)$. So we are able to reuse the analysis of the non-equilibrium diffusion model, but with elimination of the diffusion and absorption due to $\sigma_{\mathrm{s}}=O(\varepsilon^{-1})$ and $\sigma_{\mathrm{a}}=O(\varepsilon)$. Of course, the first three equations of the non-equilibrium limit are contained in both the moment model and modified moment model so the real difficulty is the last equation, i.e. E_{r} (resp. S_{r}) equation in the moment (resp. modified moment) model.

Left state: A consequence of (78) together with the hypothesis $\sigma_{\mathrm{s}}=O(\varepsilon^{-1})$ is

$$\mathbf{F}_{\mathrm{rL}}=\varepsilon\vec{v}_{\mathrm{L}}\left(\tfrac{4}{3}T_{\mathrm{rL}}^{4}\right)+O(\varepsilon^{2}). \tag{113}$$

Since we also know by a direct calculation that

$$\mathbf{F}_{\mathrm{rL}}=-\frac{4T_{\mathrm{rL}}^{4}}{3+|\vec{b}_{\mathrm{L}0}+\varepsilon\vec{b}_{\mathrm{L}1}+O(\varepsilon^{2})|^{2}}(\vec{b}_{\mathrm{L}0}+\varepsilon\vec{b}_{\mathrm{L}1}+O(\varepsilon^{2}))$$

it means that $\vec{b}_{L0}=0$ and $\vec{b}_{L1}=-\vec{v}$. It implies that n given in (98) is also $n=\frac{1}{e^{(v/\Theta_r)(1-\varepsilon\vec{v}.\vec{n}+O(\varepsilon^2))}-1}$, that is

$$n=\frac{1}{e^{(v_0/\Theta_r)(1+O(\varepsilon^2))}-1}. \tag{114}$$

We need an expansion to the first order of the entropy flux $\mathbf{Q}_{rL}$. Since the non-dimensional entropy flux is

$$\vec{Q}_{rL}=-\int\int v^2[n_L\log n_L-(n_L+1)\log(n_L+1)]\vec{n}\,dv\,d\vec{n},$$

then $\vec{Q}_{rL}$ is also

$$\begin{aligned}
&-\int\int(v_0\vec{n}_0+\varepsilon v_0\vec{v}+o(\varepsilon))[n_L\log n_L-(n_L+1)\log(n_L+1)]v_0\,dv_0\,d\vec{n}_0\\
&=-\int\int v_0^2\vec{n}_0[n_L\log n_L-(n_L+1)\log(n_L+1)]\,dv_0\,d\vec{n}_0\\
&\quad-\varepsilon\vec{v}_L\int\int v_0^2[n_L\log n_L-(n_L+1)\log(n_L+1)]\,dv_0\,d\vec{n}_0+O(\varepsilon^2).
\end{aligned}$$

By (114)

$$-\int\int v_0^2\vec{n}_0[n_L\log n_L-(n_L+1)\log(n_L+1)]\,dv_0\,d\vec{n}_0=O(\varepsilon^2).$$

We also have directly that

$$\int\int v_0^2[n_L\log n_L-(n_L+1)\log(n_L+1)]\,dv_0\,d\vec{n}_0=S_{rL}+O(\varepsilon)$$

so

$$\vec{Q}_{rL}=\varepsilon\vec{v}_L S_{rL}+O(\varepsilon). \tag{115}$$

Right state: Similarly $\mathbf{F}_{rR}=\varepsilon\vec{v}_R(\frac{4}{3}T_{rR}^4)+O(\varepsilon^2)$ and $\vec{Q}_{rR}=\varepsilon\vec{v}_R S_{rR}+O(\varepsilon)$.

Discussion of Rankine–Hugoniot relations: It is now an easy matter to deduce $O(\varepsilon)$ Rankie–Hugoniot approximations of (109) and (110). We expand the shock velocity $\sigma=\sigma_0+O(\varepsilon)$. One has

$$-\sigma_0(T_{rR}^4-T_{rL}^4)+(\tfrac{4}{3}\vec{v}_{R0}T_{rR}^4-\tfrac{4}{3}\vec{v}_{L0}T_{rL}^4)=0 \tag{116}$$

and

$$-\sigma_0(T_{rR}^3-T_{rL}^3)+(\vec{v}_{R0}T_{rR}^3-\vec{v}_{L0}T_{rL}^3)=0. \tag{117}$$

It is then clear that these Rankine–Hugoniot relations (116) and (117) are different, and that (117) is a correct approximations to the Rankine–Hugoniot relations of the Non-equilibrium limit. On the other hand, (116) is not correct. The proof now ends.

Corollary 3. *Let us specialize the previous discussion for contact discontinuities, that is $\sigma=\vec{d}.\vec{v}_L=\vec{d}.\vec{v}_R$. Then Eq.* (112) *implies that the left and right radiative temperatures are the same, $T_{rR}=T_{rL}$. On the other hand* (111) *degenerates in the sense that T_{rR} and T_{rL} are arbitrary for this equation, which is one more time compatible with the non-equilibrium model.*

The proof is straightforward. This result means that the moment model does not contain classical contact discontinuities. On the other hand, the modified moment model has these classical contact discontinuity profiles where all equations degenerate.

6. Conclusion and numerical issues

So we have justified mathematically by means of rigorous asymptotic expansions the non-equilibrium diffusion limit, already proposed in [13]. We proved that the extra term $p_{\mathrm{r}}\nabla.\vec{v}$ is a consequence of the fact that the scattering is, when isotropic, isotropic only in the comobile frame. We also prove that discontinuous solutions of standard moment model $(E_{\mathrm{r}},\mathbf{F}_{\mathrm{r}})$ are not compatible with discontinuous solution of the non-equilibrium diffusion limit. On the other hand, discontinuous solutions of the modified moment model $(S_{\mathrm{r}},\mathbf{F}_{\mathrm{r}})$ are compatible with discontinuous solution of the non-equilibrium diffusion limit.

Since modern numerical methods for the solutions of systems of conservation laws with source terms are based on Riemann solvers and shocks solutions, it is reasonable to think that the study we made about the modified moment model should help in the design of more accurate and robust conservative Eulerian schemes. But this needs to be confirmed more firmly, both theoretically and numerically.

A still open issue is the generalization of this work to moment models with frequency groups, relaxing the gray hypothesis.

Appendix A. Equivalence between the form (17) and (15) for the scattering operator S_{s}

We want to show that

$$\frac{v^2}{v_0^2}\left(\frac{1}{4\pi}\oint I_0 - I_0\right) = \frac{v_0}{v}\left(\frac{v^3}{v_0^3}\frac{1}{4\pi}\int \frac{v_0}{v'} I(v',\vec{n}')\,\mathrm{d}\vec{n}' - I\right)$$

with v' defined by

$$v' = \frac{1-\frac{\vec{n}.\vec{v}}{c}}{1-\frac{\vec{n}'.\vec{v}}{c}}v.$$

One can easily realize that the question resume to show that

$$\oint I_0\,\mathrm{d}\vec{n}_0 = \int \frac{v_0}{v'} I(v',\vec{n}')\,\mathrm{d}\vec{n}'.$$

This can be easily verified using the fact that $v\,\mathrm{d}v\,\mathrm{d}\vec{n} = v_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$ and by regularizing $\oint I_0\,\mathrm{d}\vec{n}_0$. First consider a positive and function $\phi(x)$ such that $\int_{\mathbb{R}}\phi = 1$. As a result of the theory of distribution it is well known that the sequence $\phi^{\varepsilon} = \varepsilon\phi(x/\varepsilon)$ converge toward the Dirac function when $\varepsilon \to 0$. Thus,

$$\oint I_0 = \lim_{\varepsilon\to 0}\int I_0(\bar{v},\vec{n}_0)\phi^{\varepsilon}(\bar{v}_0 - v_0)\,\mathrm{d}\vec{n}_0\,\mathrm{d}\bar{v}_0.$$

Since the measure $\nu\,\mathrm{d}\nu\,\mathrm{d}\vec{n}$ is invariant under Lorentz transform we have

$$\int I_0(\bar{\nu}_0,\vec{n}_0)\phi^\varepsilon(\bar{\nu}_0-\nu_0)\mathrm{d}\,\vec{n}_0\,\mathrm{d}\bar{\nu}_0$$

$$=\int \frac{\nu'}{\bar{\nu}_0} I_0(\bar{\nu}_0(\nu',\vec{n}'),\vec{n}_0(\nu',\vec{n}'))\phi^\varepsilon(\bar{\nu}_0(\nu',\vec{n}')-\nu_0)\,\mathrm{d}\nu'\,\mathrm{d}\vec{n}'$$

and using the invariance relation (13) for the radiative intensity we have

$$\int I_0(\bar{\nu},\vec{n}_0)\phi^\varepsilon(\bar{\nu}_0-\nu_0)\,\mathrm{d}\vec{n}_0\,\mathrm{d}\bar{\nu}_0=\int \frac{\bar{\nu}_0^2}{\nu'^2} I(\nu',\vec{n}')\phi^\varepsilon(\bar{\nu}_0(\nu',\vec{n}')-\nu_0)\,\mathrm{d}\nu\,\mathrm{d}\vec{n}'$$

with the relation $\bar{\nu}_0(\nu',\vec{n}')=\gamma\nu'(1-\vec{n}'.\vec{v}/c)$. Thus, making now the change of variables

$$\bar{\nu}_0\to\nu'=\frac{\bar{\nu}_0}{\gamma(1-\frac{\vec{n}'.\vec{v}}{c})}$$

for fixed $\vec{n}'$ we have

$$\int I_0(\bar{\nu}_0,\vec{n}_0)\phi^\varepsilon(\bar{\nu}_0-\nu_0)\,\mathrm{d}\vec{n}_0\,\mathrm{d}\bar{\nu}_0$$

$$=\int\left(\int\frac{1}{1-\vec{n}'.\vec{v}/c}\,\frac{\bar{\nu}_0^2}{\nu'^2}\,I(\nu'(\bar{\nu}_0,\vec{n}'),\vec{n}')\phi^\varepsilon(\bar{\nu}_0-\nu_0)\,\mathrm{d}\nu'\right)\mathrm{d}\vec{n}'.$$

Thus, using the fact that, in the weak sense, $\phi^\varepsilon\to\delta_0$

$$\oint I_0=\lim_{\varepsilon\to 0}\int\left(\int\frac{1}{1-\frac{\vec{n}'.\vec{v}}{c}}\,\frac{\bar{\nu}_0^2}{\nu'^2}\,I(\nu'(\bar{\nu}_0,\vec{n}'),\vec{n}')\phi^\varepsilon(\bar{\nu}_0-\nu_0)\,\mathrm{d}\nu'\right)\mathrm{d}\vec{n}'$$

$$=\int\lim_{\varepsilon\to 0}\left(\int\frac{1}{1-\frac{\vec{n}'.\vec{v}}{c}}\,\frac{\bar{\nu}_0^2}{\nu'^2}\,I(\nu'(\bar{\nu}_0,\vec{n}'),\vec{n}')\phi^\varepsilon(\bar{\nu}_0-\nu_0)\,\mathrm{d}\nu'\right)\mathrm{d}\vec{n}'$$

$$=\int\left(1-\frac{\vec{n}'.\vec{v}}{c}\right)I(\nu',\vec{n}')\,\mathrm{d}\vec{n}'=\int\frac{\nu_0}{\nu'}\,I(\nu',\vec{n}')\,\mathrm{d}\vec{n}'$$

with $\nu_0=\gamma\nu'(1-\vec{v}.\vec{n}'/c)$. Since we have also $\nu_0=\gamma\nu(1-\vec{v}.\vec{n}/c)$ thus

$$\nu'=\nu\frac{1-\frac{\vec{v}.\vec{n}}{c}}{1-\frac{\vec{v}'.\vec{n}'}{c}}.$$

Appendix B. Useful formulas and details of each terms of our P^1 model

We give here some very useful formulas for some moments integrals of a generalized Planck function. For a generalized Planck function $(15/4\pi^5)\,\nu^3/\exp\left((\nu/x)(1+\vec{y}.\vec{n})\right)-1$ we define $\mathrm{M}_1(x,\vec{y})$, $\vec{M}_2(x,\vec{y})$, $\mathbf{M}_3(x,\vec{y})$ as its moments against 1, $\vec{n}$ and $\vec{n}\otimes\vec{n}$ respectively. We recall that elementary

calculations give, see [13,17] for example:

$$\mathrm{M}_1(x,\vec{y}) = \int_{v\in[0,+\infty[} \int_{\vec{n}\in S^2} I(v,\vec{n})\,\mathrm{d}\vec{n}\,\mathrm{d}v = x^4 \frac{3+\|\vec{y}\|^2}{3(1-\|\vec{y}\|^2)^3} \tag{B.1}$$

$$\vec{M}_2(x,\vec{y}) = \int_{v\in[0,+\infty[} \int_{\vec{n}\in S^2} \vec{n} I(v,\vec{n})\,\mathrm{d}\vec{n}\,\mathrm{d}v = -\frac{4x^4\vec{y}}{3(1-\|\vec{y}\|^2)^3} \tag{B.2}$$

by setting $\vec{f} = \vec{M}_2(x,\vec{y})/\mathrm{M}_1(x,\vec{y})$ one finds that

$$\vec{y} = \left(\frac{\sqrt{4-3\|\vec{f}\|^2}-2}{\|\vec{f}\|^2}\right)\vec{f},$$

$$\mathbf{M}_3(x,\vec{y}) = \int_{v\in[0,+\infty[} \int_{\vec{n}\in S^2} \vec{n}\otimes\vec{n} I(v,\vec{n})\,\mathrm{d}\vec{n}\,\mathrm{d}v$$

$$= \left(\frac{1-\|\vec{y}\|^2}{3+\|\vec{y}\|^2}\mathbf{I} + \frac{3+\|\vec{y}\|^2}{4}\vec{f}\otimes\vec{f}\right)\mathrm{M}_1(x,\vec{y}). \tag{B.3}$$

In our P^1 model we suppose that

$$I = (15/4\pi^5)v^3/\exp\left(\frac{v}{\Theta_\mathrm{r}} + \frac{v\vec{b}.\vec{n}}{\Theta_\mathrm{r}}\right) - 1.$$

Thus, the first three moments E_r, $\vec{F}_\mathrm{r}$ and $\mathbf{P}_\mathrm{r}$ are given by

$$E_\mathrm{r} = M_1(\Theta_\mathrm{r},\vec{b}), \tag{B.4}$$

$$\vec{F}_\mathrm{r} = \vec{M}_2(\Theta_\mathrm{r},\vec{b}), \tag{B.5}$$

$$\mathbf{P}_\mathrm{r} = \mathbf{M}_3(\Theta_\mathrm{r},\vec{b}). \tag{B.6}$$

We detail now the expression for the entropy S_r and the associated flux of entropy $\vec{Q}_\mathrm{r}$. First we deal with S_r:

$$S_\mathrm{r} = -\frac{15}{4\pi^5}\int_{v\in[0,+\infty[}\int_{\vec{n}\in S^2} v^2(n\log n - (n+1)\log(n+1))\,\mathrm{d}v\,\mathrm{d}\vec{n}$$

$$= -\frac{15K}{\pi^4}\,\Theta_\mathrm{r}^3 \int_{\vec{n}} \frac{1}{(1+\vec{b}.\vec{n})^3}\,\mathrm{d}\vec{n}\int_0^{+\infty} z^2(m\log(m) - (m+1)\log(m+1)\,\mathrm{d}z \tag{B.7}$$

with $m = 1/(\exp(z)-1)$. Easy computations give

$$\int_{\vec{n}} \frac{1}{1+\vec{b}.\vec{n}}\,\mathrm{d}\vec{n} = \frac{4\pi}{(1-\|\vec{b}\|^2)^2}.$$

Now we define the function $g(\alpha)$ by

$$g(\alpha) = \int v^2(M\log M - (M+1)\log(M+1))\,\mathrm{d}v\,\mathrm{d}\vec{n}$$

with $M=(\exp(v/\alpha)-1)^{-1}$. Since $M/(M+1)=\exp(-v/\alpha)$ one has

$$g'(\alpha)=-\alpha^2\int\frac{v^4\exp(v)}{(\exp(v)-1)^2}\,\mathrm{d}v$$

and by integrating by parts

$$g'(\alpha)=-\alpha^2\int 4v^3\frac{1}{\exp(v)-1}=-4\alpha^2\frac{4\pi^5}{15}\frac{M_1(1,0)}{4\pi}=-4\alpha^2\frac{4\pi^5}{15}\frac{1}{4\pi}$$

thus

$$g(\alpha)=-\frac{4}{3}\alpha^3\frac{4\pi^5}{15}\frac{1}{4\pi}$$

and

$$\int_0^{+\infty}z^2(m\log(m)-(m+1)\log(m+1))\,\mathrm{d}z=g(1)=-\frac{4}{3}\frac{4\pi^5}{15}\frac{1}{4\pi},$$

which give finally

$$S_\mathrm{r}=\frac{4}{3}\frac{\Theta_\mathrm{r}^3}{(1-\|\vec{b}\|^2)^2}.$$

Now we compute the entropy flux. We have

$$\begin{aligned}\vec{Q}_\mathrm{r}&=-\frac{15}{4\pi^5}\int_{v\in[0,+\infty[}\int_{\vec{n}\in S^2}v^2(n\log n-(n+1)\log(n+1))\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n}\\&=-\frac{15}{4\pi^5}\Theta_\mathrm{r}^3g(1)\int\frac{\vec{n}}{(1+\vec{b}.\vec{n})^3}\,\mathrm{d}\vec{n}.\end{aligned}\tag{B.8}$$

One has now to compute $\vec{V}=\int[\vec{n}/(1+\vec{b}.\vec{n})^3]\mathrm{d}\vec{n}$. We show that $\vec{V}$ is collinear to $\vec{b}$, that is there exist a real λ such that $\vec{V}=\lambda\vec{b}$. First, we show that for every vector $\vec{b}^\perp$ percendicular to $\vec{b}$ we have $\vec{V}.\vec{b}^\perp=0$: up to a rotation, let us choose a reference such that $\vec{b}=(\|\vec{b}\|,0,0)$ and $\vec{b}^\perp=(0,x,y)$. Then we have

$$\begin{aligned}\vec{V}.\vec{b}^\perp&=\int\frac{\vec{n}.\vec{b}^\perp}{(1+\vec{b}.\vec{n})^3}\,\mathrm{d}\vec{n}\\&=\int_{\theta\in[0,\pi],\phi\in[0,2\pi]}\frac{x\sin\theta\cos\phi+y\sin\theta\sin\phi}{(1+\cos\theta\|b\|)^3}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi=0\end{aligned}$$

thus there exist λ real such that $\vec{V}=\lambda\vec{b}$. It remains now to compute λ. We have

$$\begin{aligned}\lambda\|\vec{b}\|^2&=\int\frac{\vec{n}.\vec{b}}{(1+\vec{b}.\vec{n})^3}\,\mathrm{d}\vec{n}\\&=\int_{\theta\in[0,\pi],\phi\in[0,2\pi]}\frac{\|\vec{b}\|\cos\theta\sin\theta}{(1+\cos\theta\|\vec{b}\|)^3}\,\mathrm{d}\theta=2\pi\int_{-1}^1\frac{\|\vec{b}\|x}{(1+x\|\vec{b}\|)^3}\,\mathrm{d}x\end{aligned}$$

thus

$$\vec{V}=2\pi\frac{\vec{b}}{\|\vec{b}\|}\int_{-1}^1\frac{\|\vec{b}\|x}{(1+x\|\vec{b}\|)^3}\,\mathrm{d}x$$

and one has

$$\int_{-1}^{1} \frac{\|\vec{b}\|x}{(1+x\|\vec{b}\|)^3}\,\mathrm{d}x = \frac{1}{\|\vec{b}\|}\int_{-1}^{1}\left(\frac{1}{(1+x\|\vec{b}\|)^2} - \frac{1}{(1+x\|\vec{b}\|)^3}\right)\mathrm{d}x = -\frac{2\|\vec{b}\|}{(1-\|\vec{b}\|^2)^2}$$

thus

$$\vec{Q}_\mathrm{r} = \frac{15}{4\pi^5}\Theta_\mathrm{r}^3 g(1) 2\pi \frac{\vec{b}}{\|b\|}\frac{2\|\vec{b}\|}{(1-\|\vec{b}\|^2)^2}$$

and since by the same type of calculus one obtain

$$S_\mathrm{r} = \frac{15}{4\pi^5}\,\Theta_\mathrm{r}^3 g(1)\,\frac{4\pi}{(1-\|\vec{b}\|^2)^2}$$

one has

$$\vec{Q}_\mathrm{r} = -\vec{b}S_\mathrm{r}.$$

We compute now the relaxation terms for the P^1 model. The source terms reads

$$S_E = \int_{v\in[0,+\infty[}\int_{\vec{n}\in S^2} S_\mathrm{a}(v,\vec{n}) + S_\mathrm{s}(v,\vec{n})\,\mathrm{d}v\,\mathrm{d}\vec{n} = S_E^\mathrm{a} + S_E^\mathrm{s}, \tag{B.9}$$

$$\vec{S}_F = \int_{v\in[0,+\infty[}\int_{\vec{n}\in S^2} \vec{n}(S_\mathrm{a}(v,\vec{n}) + S_\mathrm{s}(v,\vec{n}))\,\mathrm{d}v\,\mathrm{d}\vec{n} = S_F^\mathrm{a} + S_F^\mathrm{s}. \tag{B.10}$$

Let us detail each of these terms by categories. For the emission–absorption effect:

$$S_E^\mathrm{a} = \int\int \frac{v^2}{v_0^2} B(v_0,T) - \frac{v_0}{v} I\,\mathrm{d}v\,\mathrm{d}\vec{n}$$

$$= \frac{15}{4\pi^5}\int\int v^2 v_0 \left(\frac{1}{\exp\left(\frac{v_0}{T}\right)-1} - \frac{1}{\exp\left(\frac{v}{\Theta_\mathrm{r}}(1+\vec{n}.\vec{b})\right)-1}\right)\mathrm{d}v\,\mathrm{d}\vec{n}$$

and since

$$v_0 = \gamma v(1-\varepsilon\vec{n}.\vec{v})$$

$$= \frac{15}{4\pi^5}\times\int\int \gamma v^3(1-\varepsilon\vec{n}.\vec{v})\left(\frac{1}{\exp\left(\frac{v}{T}\gamma(1-\varepsilon\vec{n}.\vec{v})\right)-1} - \frac{1}{\exp\left(\frac{v}{\Theta_\mathrm{r}}(1+\vec{n}.\vec{b})\right)-1}\right)\mathrm{d}v\,\mathrm{d}\vec{n}$$

and now using (B.4) and (B.5), we obtain

$$S_E^\mathrm{a} = \gamma(M_1(T/\gamma,-\varepsilon\vec{v}) - \varepsilon\vec{v}.\vec{M}_2(T/\gamma,-\varepsilon\vec{v}) - E_\mathrm{r} + \varepsilon\vec{v}.\vec{F}_\mathrm{r}), \tag{B.11}$$

$$S_F^\mathrm{a} = \frac{15}{4\pi^5}\int\int \vec{n}v^2 v_0\left(\frac{1}{\exp\left(\frac{v_0}{T}\right)-1} - \frac{1}{\exp\left(\frac{v}{\Theta_\mathrm{r}}(1+\vec{n}.\vec{b})\right)-1}\right)\mathrm{d}v\,\mathrm{d}\vec{n}$$

and using (B.5) and (B.6), we obtain

$$S_F^\mathrm{a} = \gamma(\vec{M}_2(T/\gamma,-\varepsilon\vec{v}) - \varepsilon\mathbf{M}_3(T/\gamma,-\varepsilon\vec{v})\vec{v} - \vec{F}_\mathrm{r} + \varepsilon\mathbf{P}_\mathrm{r}\vec{v}). \tag{B.12}$$

We consider now the relaxation term due to the scattering. We recall that the scattering term we consider can be written as

$$S_{\mathrm{s}} = \frac{v^2}{v_0^2}\left(\frac{1}{4\pi}\oint I_0 - I_0\right)$$

thus

$$S_E^{\mathrm{s}} = \int S_{\mathrm{s}}\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int \frac{v^2}{v_0^2}\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\mathrm{d}v\,\mathrm{d}\vec{n}.$$

and since $v\,\mathrm{d}v\,\mathrm{d}\vec{n}$ is Lorentz-invariant measure,

$$S_E^{\mathrm{s}} = \int S_{\mathrm{s}}\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int \frac{v}{v_0}\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\mathrm{d}v_0\,\mathrm{d}\vec{n}_0.$$

Thus, using $v = \gamma v_0(1+\varepsilon\vec{v}.\vec{n}_0)$ we have

$$S_E^{\mathrm{s}} = \gamma\left(\int I_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0 - \int (1+\varepsilon\vec{v}.\vec{n}_0)I_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0\right).$$

But

$$\frac{v}{\Theta_{\mathrm{r}}}(1+\vec{b}.\vec{n}) = \frac{\gamma v_0(1+\varepsilon\vec{v}.\vec{n}_0)}{\Theta_{\mathrm{r}}} + \frac{\vec{b}}{\Theta_{\mathrm{r}}}v\vec{n}. \tag{B.13}$$

Now, for $v\vec{n}$ we call that we have

$$v_0\vec{n}_0 = v\left(n - \gamma\varepsilon\vec{v}\left(1 - \varepsilon\frac{\gamma}{\gamma+1}\vec{n}.\vec{v}\right)\right)$$
$$= (-\gamma\varepsilon\vec{v})v + \left(1 + \frac{\varepsilon\gamma^2}{\gamma+1}\vec{v}\otimes\vec{v}\right)v\vec{n}$$

or conversely

$$v\vec{n} = (\gamma\varepsilon\vec{v})v_0 + \left(1 + \frac{\varepsilon\gamma^2}{\gamma+1}\vec{v}\otimes\vec{v}\right)v_0\vec{n}_0. \tag{B.14}$$

Using (B.14) in the right-hand side of (B.13), expanding, and rearranging the term one obtains

$$\frac{v}{\Theta_{\mathrm{r}}}(1+\vec{b}.\vec{n}) = \frac{v_0}{\Theta_{\mathrm{r},0}}(1+\vec{b}.\vec{n}_0) \tag{B.15}$$

with $\Theta_{\mathrm{r},0}$ and $\vec{b}_0$ defining by

$$\Theta_{\mathrm{r},0} = \frac{\Theta_{\mathrm{r}}}{\gamma(1+\varepsilon\vec{b}.\vec{v})}$$

and

$$\vec{b}_0 = \frac{1}{1+\varepsilon\vec{b}.\vec{v}}\left(\varepsilon\vec{v} + \left(\frac{1}{\gamma}\mathbf{I} + \frac{\varepsilon\gamma}{\gamma+1}\vec{v}\otimes\vec{v}\right)\vec{b}\right)$$

which give the expression of the radiative intensity in the comobile frame

$$I_0 = \frac{15}{4\pi^5}\frac{v_0^3}{\exp\left(\frac{v_0}{\Theta_{\mathrm{r},0}}(1+\vec{b}_0.\vec{n}_0)\right) - 1}.$$

Now it is simple to compute the relaxation terms due to the scattering

$$S_E^{\mathrm{s}} = \gamma(\mathrm{M}_1(\Theta_{\mathrm{r},0},\vec{b}_0) - \mathrm{M}_1(\Theta_{\mathrm{r},0},\vec{b}_0) - \varepsilon\vec{v}.\vec{M}_2(\Theta_{\mathrm{r},0},\vec{b}_0))$$
$$= -\gamma\varepsilon\vec{v}.\vec{M}_2(\Theta_{\mathrm{r},0},\vec{b}_0) \tag{B.16}$$

and

$$S_F^{\mathrm{s}} = \int S_{\mathrm{s}}\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n} = \int \frac{v^2}{v_0^2}\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\vec{n}\,\mathrm{d}v\,\mathrm{d}\vec{n}$$
$$= \int \frac{v}{v_0}\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\vec{n}_0\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$$

and using one more time the relation (B.14) for $v\vec{n}$ we obtain

$$S_F^{\mathrm{s}} = \int \left(\gamma\varepsilon\vec{v} + \left(1 + \frac{\varepsilon\gamma^2}{\gamma+1}\,\vec{v}\otimes\vec{v}\right)\vec{n}_0\right)\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\vec{n}\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0$$
$$= \left(\mathbf{I} + \frac{\varepsilon\gamma^2}{\gamma+1}\,\vec{v}\otimes\vec{v}\right)\int \vec{n}_0\left(\frac{1}{4\pi}\oint I_0 - I_0\right)\vec{n}\,\mathrm{d}v_0\,\mathrm{d}\vec{n}_0.$$

Finally, using (B.2) we find that

$$S_F^{\mathrm{s}} = -\left(\mathbf{I} + \frac{\varepsilon\gamma^2}{\gamma+1}\,\vec{v}\otimes\vec{v}\right)\vec{M}_2(\Theta_{\mathrm{r},0},\vec{b}_0). \tag{B.17}$$

We are also able to compute the relaxation term for S_{r} we recall that we have

$$S_{S_{\mathrm{r}}} = \frac{S_E + \vec{b}.S_F}{\Theta_{\mathrm{r}}}.$$

By example for the scattering this gives

$$S_{S_{\mathrm{r}}}^{\mathrm{s}} = -\frac{1}{\Theta_{\mathrm{r}}}\left(\vec{M}_2(\Theta_{\mathrm{r},0},\vec{b}_0).\left(\gamma\varepsilon\vec{v} + \left(\mathbf{I} + \frac{\varepsilon\gamma^2}{\gamma+1}\,\vec{v}\otimes\vec{v}\right)\vec{b}\right)\right)$$

but

$$\left(\gamma\varepsilon\vec{v} + \left(\mathbf{I} + \frac{\varepsilon\gamma^2}{\gamma+1}\,\vec{v}\otimes\vec{v}\right)\vec{b}\right) = \gamma(1+\varepsilon\vec{b}.\vec{v})\vec{b}_0$$

thus

$$S_{S_{\mathrm{r}}}^{\mathrm{s}} = -\frac{\gamma(1+\varepsilon\vec{b}.\vec{v})}{\Theta_{\mathrm{r}}}(\vec{b}_0.\vec{M}_2(\Theta_{\mathrm{r},0},\vec{b}_0)).$$

and using (B.2) and the definition of $\Theta_{\mathrm{r},0}$ we have finally

$$S_{S_{\mathrm{r}}}^{\mathrm{s}} = \frac{4}{3}\,\frac{\|\vec{b}_0\|^2}{(1-\|\vec{b}_0\|^2)^2}\,\Theta_{\mathrm{r},0}^3. \tag{B.18}$$

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