On the Chang and Cooper numerical scheme applied to a linear Fokker-Planck equation

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Abstract

In this article, we show that for a particular linear Fokker-Planck operator, the explicit Chang and Cooper scheme preserves the positivity of the distribution and allows the distribution to converges toward the thermodynamical equilibrium by preserving the decreasing of the entropy under a classical CFL criteria.

Introduction

The Chang and Cooper scheme (cf. [1]) is a classical scheme (cf. [2], [3] and [4]) used to solve a kinetic equation of the type

$$\partial_t f = S(f)$$

where $S(f)$ is a Fokker-Planck operator. The main property of the Chang and Cooper scheme is that, at least in the linear case, the numerical fluxes are equal to zero when the distribution f is equal to the equilibrium distribution: in other words, this scheme preserves the thermodynamical equilibrium when it is reached.

Although, up to now, it exists no convergence properties in large time toward this equilibrium. Then, we will show in that article that for a particular linear Fokker-Planck operator, the explicit Chang and Cooper scheme has good convergence properties.

The Fokker-Planck operator studed in that paper is defined by

$$S(f)(v) = \Omega \nabla_v \cdot \left[(\overrightarrow{v} - \overrightarrow{U_e}) f(v) + \frac{T_e}{m} \nabla_v f \right] \quad (1)$$

for the convection-diffusion form; $\overrightarrow{U_e}$, $T_e > 0$ (which respectively are the velocity and the temperature of the medium), $m > 0$ and $\Omega > 0$ (which respectively are the atomic mass of the particle and the collision frequency

of the particle on the medium) are constants. Let us note that this operator is the linear version of the non linear ion-electron collision operator studied in [5] in which Ω, $\overrightarrow{U_e}$ and T_e depends upon the time. By defining the equilibrium distribution

$$\mathcal{M}_{N,\overrightarrow{U_e},T_e}(v) = \frac{N}{(2\pi T_e/m)^{3/2}} \exp\left[-\frac{m(\overrightarrow{v}-\overrightarrow{U_e})^2}{2T_e}\right],$$

we can also define $S(f)$ with the two equivalent forms that is to say with

$$S(f) = \Omega\frac{T_e}{m}\nabla_v \cdot [f\nabla_v \log(f/\mathcal{M}_{N,\overrightarrow{U_e},T_e})]$$

which is called the Landau form, and with

$$S(f) = \Omega\frac{T_e}{m}\nabla_v \cdot [\mathcal{M}_{N,\overrightarrow{U_e},T_e}\nabla_v(f/\mathcal{M}_{N,\overrightarrow{U_e},T_e})] \tag{2}$$

which is called the non logarithmic Landau form. The explicit Chang and Cooper scheme (cf. [1]) is built in order to make the discretised fluxes $(\overrightarrow{v}-\overrightarrow{U_e})f(v) + \frac{T_e}{m}\nabla_v f$ equal to zero when $f = \mathcal{M}_{N,\overrightarrow{U_e},T_e}$ which implies that this scheme preserves the thermodynamical equilibrium when it is reached.

In this article, we will show that by using the explicit Chang and Cooper numerical scheme to discretize the Fokker-Planck operator (1), the distribution f converges toward the thermodynamical equilibrium $\mathcal{M}_{N,\overrightarrow{U_e},T_e}$ in large time under a classical CFL criteria. For a seak of simplicity, we define the Fokker-Planck operator $S(f)$ in cartesian geometry, the microscopic velocity having only one dimension. Then, we replace now $(2\pi T_e/m)^{3/2}$ with $\sqrt{2\pi T_e/m}$ and ∇_v with ∂_v.

The velocity space is discretized with the series (v_j) where $j \in \{1, ..., j_{\max}\}$; the velocity step is constant and equal to Δv and finally, we define

$$< g > \equiv \sum_j g(v_j)\Delta v.$$

The time subscript is n and the time step is defined with Δt.

1 The explicit Chang and Cooper scheme

The explicit Chang and Cooper scheme is defined by

$$\frac{1}{\Delta t}(f_j^{n+1} - f_j^n) = S(f^n)_j \tag{3}$$

with

$$\begin{aligned} S(f^n)_j = \\ \frac{\Omega}{\Delta v}\left[(v_{j+1/2}-U_e)\widetilde{f}^n_{j+1/2} - (v_{j-1/2}-U_e)\widetilde{f}^n_{j-1/2}\right] \\ +\frac{\Omega T_e}{m\Delta v^2}(a_j f^n_{j+1} - b_j f^n_j + c_j f^n_{j-1}) \end{aligned} \tag{4}$$

where $a_j = c_j = 1$ et $b_j = 2$ except at the frontier of the velocity domain (see below). $\widetilde{f}^n_{j+1/2}$ is an approximation of $f(t = t_n, v = v_{j+1/2})$ defined with the following definition (cf. [1]):

Definition *The Chang and Cooper average* $\widetilde{f}_{j+1/2}$ *of the quantities* f_j *and* f_{j+1} *is defined by*

$$\widetilde{f}_{j+1/2} = \delta_{j+1/2} f_j + (1 - \delta_{j+1/2}) f_{j+1}$$

with

$$\delta_{j+1/2} = \frac{1}{w_{j+1/2}} - \frac{1}{\exp(w_{j+1/2}) - 1}$$

where

$$w_{j+1/2} = \frac{m\Delta v}{T_e}(v_{j+1/2} - U_e).$$

Boundary conditions and the mass conservation To make the scheme conservative, we have to impose boundary conditions of the type Robin that is to say we impose on the boundary velocity domain

$$(v - U_e)f + \frac{T_e}{m}\partial_v f = 0$$

which is equivalent to define for the numerical scheme

$$\begin{cases} a_j = 1 \text{ si } j \neq j_{\max}, \\ b_j = 2 \text{ si } j \in \{2, ..., j_{\max} - 1\}, \\ c_j = 1 \text{ si } j \neq 1, \\ b_1 = b_{j_{\max}} = 1 \text{ and } a_{j_{\max}} = c_1 = 0 \end{cases} \tag{5}$$

and

$$\widetilde{f}_{1/2} = \widetilde{f}_{j_{\max}+1/2} \equiv 0. \tag{6}$$

We have the following conservation property:

Property 1.1

$$< f_j^{n+1} > = < f_j^n > .$$

Let us now introduce the following notations:

Notation We define

$$\widehat{\mathcal{M}}_{f^0,j+1/2} = \frac{\mathcal{M}_{f^0,j+1}\mathcal{M}_{f^0,j}}{\widetilde{\mathcal{M}}_{f^0,j+1/2}}$$

where $\widetilde{\mathcal{M}}_{f^0,j+1/2}$ is the *entropic average* of $\mathcal{M}_{f^0,j+1}$ and of $\mathcal{M}_{f^0,j}$ that is to say

$$\widetilde{\mathcal{M}}_{f^0,j+1/2} = \frac{\mathcal{M}_{f^0,j+1} - \mathcal{M}_{f^0,j}}{\log \mathcal{M}_{f^0,j+1} - \log \mathcal{M}_{f^0,j}}.$$

The entropic average was firstly introduce to discretize the ion-electron collision operator (see [5]) and the isotropic ion-ion collision operator (see [6]; see also the first part of [7] for more details). Now, we establish the following property and lemma:

Property 1.2 *When* $\widetilde{f}^n_{j+1/2}$ *is the Chang and Cooper average of* f_j^n *and of* f_{j+1}^n*, we can define* $\widetilde{f}^n_{j+1/2}$ *in the following way*

$$\widetilde{f}^n_{j+1/2} = \frac{f^n_{j+1} - f^n_j}{\log \mathcal{M}_{f^0,j+1} - \log \mathcal{M}_{f^0,T_e,j}} \tag{7}$$

$$+ \left(\frac{f_j^n}{\mathcal{M}_{f^0,j}} - \frac{f^n_{j+1}}{\mathcal{M}_{f^0,j+1}}\right) \cdot \frac{\mathcal{M}_{f^0,j+1}\mathcal{M}_{f^0,j}}{\mathcal{M}_{f^0,j+1} - \mathcal{M}_{f^0,j}}.$$

Lemma 1.1 *When $\widetilde{f}^n_{j+1/2}$ is the Chang and Cooper average of f^n_j and of f^n_{j+1}, the operator $S(f^n)_j$ defined with (4) can be written with*

$$S(f^n)_j =$$

$$\frac{\Omega T_e}{m\Delta v^2}\Big\{\widehat{\mathcal{M}}_{f^0,j+1/2}[(f^n/\mathcal{M}_{f^0})_{j+1}-(f^n/\mathcal{M}_{f^0})_j] \tag{8}$$

$$-\widehat{\mathcal{M}}_{f^0,j-1/2}[(f^n/\mathcal{M}_{f^0})_j-(f^n/\mathcal{M}_{f^0})_{j-1}]\Big\}$$

where the boundary conditions (5) and (6) are replaced with the boundary conditions

$$f^n_0 \equiv f^n_1\cdot\frac{\mathcal{M}_{f^0,0}}{\mathcal{M}_{f^0,1}} \quad \textit{and} \quad f^n_{j_{\max}+1}\equiv f^n_{j_{\max}}\cdot\frac{\mathcal{M}_{f^0,j_{\max}+1}}{\mathcal{M}_{f^0,j_{\max}}}. \tag{9}$$

In other words, the Chang and Cooper average makes equivalent from a discretized point of view the convection-diffusion form (1) and the non logarithmic Landau form (2). More over, let us remark that the boundary conditions (9) are equivalent from a discretized point of view to the Robin type boundary conditions

$$\partial_v(f/\mathcal{M}_{f^0})=0.$$

Proof of the property 1.2 We easily verify that

$$w_{j+1/2}\equiv\frac{m\Delta v}{T_e}(v_{j+1/2}-U_e)=-\left[(\log\mathcal{M}_{f^0})_{j+1}-(\log\mathcal{M}_{f^0})_j\right]. \tag{10}$$

Then

$$\delta_{j+1/2}=-\frac{1}{\log\mathcal{M}_{f^0,j+1}-\log\mathcal{M}_{f^0,j}}+\frac{\mathcal{M}_{f^0,j+1}}{\mathcal{M}_{f^0,j+1}-\mathcal{M}_{f^0,j}}.$$

Then

$$\widetilde{f}^n_{j+1/2}=\frac{f^n_{j+1}-f^n_j}{\log\mathcal{M}_{f^0,j+1}-\log\mathcal{M}_{f^0,j}}+\mathcal{M}_{f^0,T_e,j+1}\cdot\frac{f^n_j-f^n_{j+1}}{\mathcal{M}_{f^0,j+1}-\mathcal{M}_{f^0,j}}+f^n_{j+1}$$

which shows that

$$\widetilde{f}^n_{j+1/2}=\frac{f^n_{j+1}-f^n_j}{\log\mathcal{M}_{f^0,j+1}-\log\mathcal{M}_{f^0,j}}+\left(\frac{f^n_j}{\mathcal{M}_{f^0,j}}-\frac{f^n_{j+1}}{\mathcal{M}_{f^0,j+1}}\right)\cdot\frac{\mathcal{M}_{f^0,j+1}\mathcal{M}_{f^0,j}}{\mathcal{M}_{f^0,j+1}-\mathcal{M}_{f^0,j}}.$$

□

Proof of the lemma 1.1 By using (7), we obtain that when $j\notin\{1,j_{\max}\}$

$$\begin{aligned}S(f^n)_j=&\frac{\Omega}{\Delta v}\left(\frac{f^n_j}{\mathcal{M}_{f^0,j}}-\frac{f^n_{j+1}}{\mathcal{M}_{f^0,j+1}}\right)\cdot\frac{\mathcal{M}_{f^0,j+1}\mathcal{M}_{f^0,j}}{\mathcal{M}_{f^0,j+1}-\mathcal{M}_{f^0,j}}\left(v_{j+1/2}-U_e\right)\\&-\frac{\Omega}{\Delta v}\left(\frac{f^n_{j-1}}{\mathcal{M}_{f^0,j-1}}-\frac{f^n_j}{\mathcal{M}_{f^0,j}}\right)\cdot\frac{\mathcal{M}_{f^0,j}\mathcal{M}_{f^0,j-1}}{\mathcal{M}_{f^0,j}-\mathcal{M}_{f^0,j-1}}\left(v_{j-1/2}-U_e\right)\\&+\frac{\Omega}{\Delta v}\left[\frac{f^n_{j+1}-f^n_j}{\log\mathcal{M}_{f^0,j+1}-\log\mathcal{M}_{f^0,j}}\left(v_{j+1/2}-U_e\right)-\frac{f^n_j-f^n_{j-1}}{\log\mathcal{M}_{f^0,j}-\log\mathcal{M}_{f^0,j-1}}\left(v_{j-1/2}-U_e\right)\right]\\&+\frac{\Omega T_e}{m\Delta v^2}\left(f^n_{j+1}-2f^n_j+f^n_{j-1}\right).\end{aligned}$$

But, by taking into account (10), we can write that

$$\frac{\mathcal{M}_{f^0,j+1}\mathcal{M}_{f^0,j}}{\mathcal{M}_{f^0,j+1}-\mathcal{M}_{f^0,j}}\left(v_{j+1/2}-U_e\right)=-\frac{T_e}{m\Delta v}\widehat{\mathcal{M}}_{f^0,j+1/2}$$

and that

$$\frac{(v_{j+1/2}-U_e)}{\log \mathcal{M}_{f^0,j+1}-\log \mathcal{M}_{f^0,j}} = -\frac{T_e}{m\Delta v}.$$

Then, we have

$$\begin{aligned} S(f^n)_j = \frac{\Omega T_e}{m\Delta v^2}\Bigg[&\left(\frac{f^n_{j+1}}{\mathcal{M}_{f^0,j+1}}-\frac{f^n_j}{\mathcal{M}_{f^0,j}}\right)\cdot\widehat{\mathcal{M}}_{f^0,j+1/2} \\ -&\left(\frac{f^n_j}{\mathcal{M}_{f^0,j}}-\frac{f^n_{j-1}}{\mathcal{M}_{f^0,j-1}}\right)\cdot\widehat{\mathcal{M}}_{f^0,j-1/2}\Bigg] \\ &-\frac{\Omega T_e}{m\Delta v^2}\left(f^n_{j+1}-2f^n_j+f^n_{j-1}\right) \\ &+\frac{\Omega T_e}{m\Delta v^2}\left(f^n_{j+1}-2f^n_j+f^n_{j-1}\right) \end{aligned}$$

that is to say

$$\begin{aligned} S(f^n)_j = \frac{\Omega T_e}{m\Delta v^2}\Big\{&\widehat{\mathcal{M}}_{f^0,j+1/2}[(f^n/\mathcal{M}_{f^0})_{j+1}-(f^n/\mathcal{M}_{f^0})_j] \\ &-\widehat{\mathcal{M}}_{f^0,j-1/2}[(f^n/\mathcal{M}_{f^0})_j-(f^n/\mathcal{M}_{f^0})_{j-1}]\Big\}. \end{aligned}$$

If $j\in\{1,j_{\max}\}$, we easily verify that the equality (8) remains true when

$$f^n_0 \equiv f^n_1\cdot\frac{\mathcal{M}_{f^0,0}}{\mathcal{M}_{f^0,1}} \quad \text{and} \quad f^n_{j_{\max}+1}\equiv f^n_{j_{\max}}\cdot\frac{\mathcal{M}_{f^0,j_{\max}+1}}{\mathcal{M}_{f^0,j_{\max}}}.$$

□

2 Positivity of the scheme

Let us introduce some new notations:

Notation Let us define

$$\Delta t^n = \Delta t^0_1\cdot\frac{h^n_{\min}}{h^n_{\max}} \qquad \text{where} \qquad \Delta t^0_1 = \frac{m}{4\Omega T_e}\cdot\frac{\Delta v^2}{M^0},$$

$$\left\{\begin{array}{l} h^n_{\max} = \max_j\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j, \\ \\ h^n_{\min} = \min_j\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j, \end{array}\right.$$

$$\mathcal{M}_{f^0} = \frac{N^0}{\sqrt{2\pi T_e/m}}\exp[-\frac{m(v-U_e)^2}{2T_e}],$$

$$M^0 = \max_j\left(\frac{\mathcal{M}_{f^0,j\pm1}}{\mathcal{M}_{f^0,j}}\right)$$

and

$$H^n = <\left[f^n\log\left(f^n/\mathcal{M}_{f^0}\right)\right]_j>.$$

We have the following proposition:

Proposition 2.1 *For all strictly positive initial condition, when $\widetilde{f}^n_{j+1/2}$ is the Chang and Cooper average of f^n_j and of f^n_{j+1}, the explicit scheme defined with (3) and with (4) verifies the inequality*

$$h^n_{\min}\le h^{n+1}_{\min}\le h^{n+1}_{\max}\le h^n_{\max}$$

when

$$\Delta t < 2\Delta t_1^0.$$

More over, the time step Δt^n *verifies*

$$\Delta t^n \leq \Delta t^{n+1} \leq \Delta t_1^0.$$

Let us note that this proposition implies that

$$\inf_{j,n} f_j^n > 0$$

and allows to establish that the time step will never be equal to zero. Then, the numerical scheme preserves the positivity of the distribution under a CFL criteria, it exists $h_{\min}^\infty$ and $h_{\max}^\infty$ such that the series $(h_{\min}^n)$ and $(h_{\max}^n)$ admits the respective limits $h_{\min}^\infty$ and $h_{\max}^\infty$ when n goes to $+\infty$. But, we do not have still proove that $h_{\min}^\infty = h_{\max}^\infty$, equality which would imply that it would exist a constant $C > 0$ such that

$$\lim_{n\to\infty} \left(f_j^n\right) = C \cdot \left(\mathcal{M}_{f^0,j}\right).$$

Proof of the proposition 2.1 By defining $h_j^n = f_j^n / \mathcal{M}_{f^0,j}$ and by using the lemma 1.1, we can say that

$$f_j^{n+1} = f_j^n + \frac{\Delta t \Omega T_e}{m \Delta v^2} \left[\widehat{\mathcal{M}}_{f^0,j+1/2}\left(h_{j+1}^n - h_j^n\right) + \widehat{\mathcal{M}}_{f^0,j-1/2}\left(h_{j-1}^n - h_j^n\right)\right].$$

Then, it is obvious that

$$h_j^n - \frac{\Delta t \Omega T_e}{m\Delta v^2} \cdot \frac{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}}{\mathcal{M}_{f^0,j}} \left(h_j^n - h_{\min}^n\right) \leq h_j^{n+1}$$

and that

$$h_j^{n+1} \leq h_j^n + \frac{\Delta t \Omega T_e}{m\Delta v^2} \cdot \frac{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}}{\mathcal{M}_{f^0,j}} \left(h_{\max}^n - h_j^n\right).$$

Let us suppose that Δt is such that

$$\forall j, \Delta t \leq \frac{m\Delta v^2}{\Omega T_e} \cdot \frac{\mathcal{M}_{f^0,j}}{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}}.$$

Then, we obtain that

$$\forall j : h_{\min}^n \leq h_j^{n+1} \leq h_{\max}^n.$$

More over, we have

$$\frac{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}}{\mathcal{M}_{f^0,j}} = \frac{\log\left(\mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j+1}\right)}{\mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j+1} - 1} + \frac{\log\left(\mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j-1}\right)}{\mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j-1} - 1}.$$

Then

$$\frac{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}}{\mathcal{M}_{f^0,j}} \leq \frac{1}{\min(1, \mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j+1})} + \frac{1}{\min(1, \mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j-1})}$$

since $\forall x \geq 0 : \min(1,x) \leq (x-1)/\log x$. But

$$\begin{aligned}
\frac{1}{\min(1, \mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j+1})} + \frac{1}{\min(1, \mathcal{M}_{f^0,j}/\mathcal{M}_{f^0,j-1})} &\leq \frac{2}{\min_k(\mathcal{M}_{f^0,k\pm1}/\mathcal{M}_{f^0,k})} \\
&= 2\max_k(\mathcal{M}_{f^0,k\pm1}/\mathcal{M}_{f^0,k}) \\
&= 2M^0.
\end{aligned}$$

Finally, we can then write that

$$\forall j : \frac{1}{2M^0} \leq \frac{\mathcal{M}_{f^0,j}}{\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}} \tag{11}$$

which allows to state that

$$\Delta t \leq 2\Delta t_1^0 \rightarrow h_{\min}^n \leq h_{\min}^{n+1} \leq h_{\max}^{n+1} \leq h_{\max}^n.$$

More over, it is obvious that

$$h_{\min}^n \leq h_{\min}^{n+1} \leq h_{\max}^{n+1} \leq h_{\max}^n \Longrightarrow \Delta t^n \leq \Delta t^{n+1}.$$

□

3 Convergence toward the thermodynamical equilibrium

The following proposition shows that the distribution f^n converges toward the thermodynamical equilibrium:

Proposition 3.1 *For all strictly positive initial condition, when $\widetilde{f}_{j+1/2}^n$ is the Chang and Cooper average of f_j^n and of f_{j+1}^n and when*

$$\Delta t < \Delta t^n,$$

the explicit scheme defined with (3) and with (4) verifies the entropic inequality

$$H^{n+1} \leq H^n$$

and

$$\lim_{t^n \to +\infty} \left(f_j^n\right) = \frac{N^0}{< \mathcal{M}_{f^0,j} >} \cdot \left(\mathcal{M}_{f^0,j}\right)$$

Then, we have

$$\lim_{t^n \to +\infty} \Delta t^n = \Delta t_1^0.$$

Proof of the proposition 3.1 Since $\Delta t < \Delta t_1^0$, we have $f_j^{n+1} > 0$ by using the proposition 2.1: then, we can evaluate H^{n+1}. And, by using the inequality

$$\forall x > 0 : \log(x+1) < x,$$

we easily obtain that

$$H^{n+1} \leq H^n + \Delta t \sum_j \left[S(f^n)_j \log\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j + \Delta t \frac{S(f^n)_j^2}{f_j^n} \right] \Delta v. \tag{12}$$

More over, by applying the Schwarz's inequality, we obtain that

$$\begin{aligned} S(f^n)_j^2 \quad \leq \quad & \frac{\Omega T_e}{m\Delta v^2} \left(\widehat{\mathcal{M}}_{f^0,j+1/2} + \widehat{\mathcal{M}}_{f^0,j-1/2}\right) \cdot \\ & \frac{\Omega T_e}{m\Delta v^2} \left[\widehat{\mathcal{M}}_{f^0,j+1/2} \left(h_{j+1}^n - h_j^n\right)^2 + \widehat{\mathcal{M}}_{f^0,j-1/2} \left(h_{j-1}^n - h_j^n\right)^2\right] \end{aligned}$$

where we have define $h_j^n = f_j^n / \mathcal{M}_{f^0,j}$. And, by using (11), we obtain

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq \frac{\Omega T_e}{m\Delta v^2} \cdot \frac{2M^0}{h_{\min}^n} \cdot \frac{2\Omega T_e}{m\Delta v^2} \sum_j \widehat{\mathcal{M}}_{f^0,j+1/2} \left(h_{j+1}^n - h_j^n\right)^2.$$

And since

$$\forall x \geq 0, \forall y \geq 0 : \frac{x-y}{\log(x/y)} \leq \max(x,y),$$

we have

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq \frac{\Omega T_e}{m\Delta v^2} \cdot \frac{2M^0}{h^n_{\min}} \cdot \frac{2\Omega T_e}{m\Delta v^2} \sum_j \widehat{\mathcal{M}}_{f^0,j+1/2} \left(h^n_{j+1} - h^n_j\right) \cdot \left(\log h^n_{j+1} - \log h^n_j\right) h^n_{\max}.$$

More over, we have

$$\sum_j S(f^n)_j \cdot \log(f^n/\mathcal{M}_{f^0})_j =$$

$$\frac{\Omega T_e}{m\Delta v^2} \sum_j \widehat{\mathcal{M}}_{f^0,j+1/2} [(f^n/\mathcal{M}_{f^0})_{j+1} - (f^n/\mathcal{M}_{f^0})_j] \log(f^n/\mathcal{M}_{f^0})_j$$
$$-\frac{\Omega T_e}{m\Delta v^2} \sum_j \widehat{\mathcal{M}}_{f^0,j-1/2} [(f^n/\mathcal{M}_{f^0})_j - (f^n/\mathcal{M}_{f^0})_{j-1}] \log(f^n/\mathcal{M}_{f^0})_j$$

that is to say

$$\sum_j S(f^n)_j \cdot \log(f^n/\mathcal{M}_{f^0})_j = \tag{13}$$
$$-\frac{\Omega T_e}{m\Delta v^2} \sum_j \widehat{\mathcal{M}}_{f^0,j+1/2} \left(h_{j+1} - h_j\right) \cdot \left(\log h^n_{j+1} - \log h^n_j\right) \leq 0$$

by using the convexity property of $x \mapsto \log x$. Then, we can write that

$$\sum_j \frac{S(f^n)_j^2}{f_j^n} \leq -\frac{4\Omega T_e}{m\Delta v^2} \cdot \frac{h^n_{\max}}{h^n_{\min}} M^0 \sum_j S(f^n)_j \log\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j.$$

Finally, we obtain

$$H^{n+1} \leq H^n + \Delta t \left(1 - \frac{4\Delta t \Omega T_e}{m\Delta v^2} \cdot M^0 h^n_{\max}/h^n_{\min}\right) \cdot \sum_j S(f^n)_j \log\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j \Delta v.$$

Then, when $\Delta t < \Delta t^n$, by using the inequality (13), we obtain the inequality

$$H^{n+1} \leq H^n + \Delta t \left(1 - \frac{\Delta t}{\Delta t^n}\right) \cdot \sum_j S(f^n)_j \log\left(\frac{f^n}{\mathcal{M}_{f^0}}\right)_j \Delta v \leq H^n. \tag{14}$$

(...)$\square$

Conclusion

In that paper, we have shown that for a particular Fokker-Planck linear operator, the explicit Chang and Cooper scheme has very good properties: under a classical CFL criteria, it preserves the positivity of the distribution, the entropy decreases and the distribution converges toward the thermodynamical equilibrium.

This theoretical results are similar to those obtained with this linear Fokker-Planck operator by replacing the Chang and Cooper average with the entropic average introduced in [5]. But, we have to focus on the fact that all these results are obtained in the linear case.

Although, by using the entropic average, it is possible to obtain similar theoretical results in the non linear case (cf. [5]) which is not at all the case when we use the Chang and Cooper average (cf. [7], first part). This is underlined by the fact that when it is simulated ion-electron collisions under very hard conditions (as those met in the field of the Inertial Confinement Fusion: see the last chapter of the second part of [7]) with the non linear version of the linear Fokker-Planck operator discretized in that paper, it is possible to go to the end of the simulation without any computer error only by using the entropic average. Then, it seems that it will be very difficult to obtain good theoretical results in the non linear case for the Chang and Cooper average.

References

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