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An asymptotic preserving scheme for Hydrodynamics Radiative Transfert Models

Numerical Scheme for radiative transfert

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Summary In this paper, we shall propose a numerical scheme consisting of two steps: the first based relaxation method and the second on the so called well balanced scheme. The derivation of the scheme relies on the resolution of the stationnary Riemann problem with source terms. The obtained scheme is compatible with the diffusive regime of hydrodynamics radiative transfert models. Some numerical results are shown.

1 Introduction

In this article , we are concerned with a model arising from radiative transfert modelling. It is well known that in very rarefied regions, the physically relevant model is a free transport one whereas in the dense regions, the radiative transfert becomes a diffusion equation. The aim of this work is to design a scheme for the two-moment systems that can be obtain, for example using maximum entropy technics. We refer to [23] for a recent presentation of the various closures.

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The requierement for the scheme are to be used with a non uniform grid in space, to deal with varying scattering cross sections, to have the correct asymptotics behabiour in diffusive regimes, to be implicit in order to avoid too restrictive time steps limitations.

The solution we should described has been announced in [5] and it consists of two main steps : the first one is a relaxation method which permits to transform the nonlinear hyperbolic system into two independant linear systems, known as the Goldstein Taylor system or Telegraph equations; the second step is to use the so called well balanced scheme for each of the two systems. Moreover, we shall propose a new interpretation of the well balanced schemes as a Godunov scheme when dealing with source terms. This interpretation is more convenient in order to extend the well balanced schemes for multi dimensional problem.

The paper is organized as follows : in section 2, we recall the model of interest and its main properties, namely an invariant domain or equivalently the positivity of some quantities which are of great physical importance and the diffusive asymptotics i.e. regimes where the solutions behave like solution of a parabolic equation. In section 3, we describe the proposed numerical scheme and its derivation in two steps (relaxation method in subsection 3.1 and well balanced scheme in subsection 3.2). In section 4, we present two numerical results : the first one is concerned with a varying cross section and the second one to a coupled system with a heat equation for the temperature of material.

2 Radiative transfert hydrodynamical models

The models, we are interested in, arise from the radiation transport equation, which is a kinetic equation for the specific intensity of photons $I(\Omega, \nu, r, t)$ after integration over the angular variable Ω and the frequency ν. We refer to [18,6] for a detailled presentation.

In this paper, we are concerned with systems of conservation laws for the two first moments of the intensity, namely $(\rho, \rho u)$:

$$\begin{cases} \partial_t \rho + \nabla_x(\rho u) \ = \ 0 \\ \partial_t(\rho u) + \nabla_x \cdot P = 0 \ , \end{cases} \tag{1}$$

where P is the pressure tensor of the form

$$P = \rho \left(h(|u|) I_d + (1 - 3h(|u|)) \frac{\mathbf{u} \otimes \mathbf{u}}{\|u\|^2} \right) \ ,$$

where $h(x) = \frac{x}{g^{-1}(x)}$, $x > 0$ with

$$g(x) = \coth(x) - \frac{1}{x} \ ,$$

the so called Langevin function.

2.1 Eddington factors

In monodimensional case (the velocity is parallel to the first axis) the above system reduces to

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) &= \quad 0 \\ \partial_t(\rho u) + \partial_x \rho h(u) &= -\sigma \rho u \ , \end{cases} \tag{2}$$

where the function h has several forms, which are called Eddington factors. We refer to [18] or to [23] for a detailled presentation. For example, Kershaw suggests,

$$h(x) = \frac{1 + 2x^2}{3}. \tag{3}$$

Minerbo uses entropy arguments to obtain the Eddington factor ,

$$h(x) = 1 - \frac{2x}{g^{-1}(x)}, \tag{4}$$

which was called the maximum entropy Eddington factor.

Also, Minerbo [19] suggested that any intensity may be approximated as a linear function, so the Eddington factor Minerbo linear is:

$$h(x) = \begin{cases} \frac{1}{3} \ , x \in \left[0, \frac{1}{3}\right] \ , \\ \frac{1}{2}(1-x)^2 + x^2 \ , x \in \left(\frac{1}{3}, 1\right] . \end{cases} \tag{5}$$

Using a Chapman-Enskog approach, we have the following Eddington factor sugested by Levermore :

$$h(x) = \coth x \left(\coth x - \frac{1}{x} \right) .. \tag{6}$$

Another Eddington factor is Levermore-Lorentz [19,18]:

$$h(x) = \frac{1}{3} + \frac{2x^2}{2 + \sqrt{4 - 3x^2}} \ . \tag{7}$$

Let us propose another function, with the same properties of Eddington factors, called Minerbo rational, which is defined by:

$$h(x) = \frac{1}{3} + \frac{2x^2}{5 - |x| - x^2} \ . \tag{8}$$

This Eddington factors is obtained by making some assumptions for the function

$$h(x) = \frac{1}{3} + \frac{a|x| + b}{cx^2 + d|x| + e} \ , \tag{9}$$

namely:

$$\begin{aligned} h(0) &= \tfrac{1}{3} \ , \\ h'(0) &= 0 \ , \\ h''(0) &= \tfrac{2}{5} \ . \end{aligned}$$

and

$$\begin{aligned} h(1) &= 1 \ , \\ h'(1) &= 2 \ . \end{aligned}$$

We remark that these Eddington factors are increasing and convexe functions.Let us assume the following hypothesis on the h function : h is a increasing, convex function

$$u^2 < h(u) \le 1 \ , h(0) = \frac{1}{3} \ , h(1) = 1 \ . \tag{10}$$

Remark 1 Let us mention that for $h \equiv 1$, the system (2) is known as the Goldstein-Taylor equation or as the telegraph equation. This particular system will be used in the construction of the solution of the nonlinear system of interest.

Remark 2 Note also that some two dimensional closure have been proposed [24] where the pressure is a function of ρ and j separately instead of a function of the form $\rho h(j/\rho)$ as in (2).

2.2 Invariant domain

Let us now give some properties of the solutions of the nonlinear hyperbolic system (2).

First, for any physically admissible state (ρ, j) such that

$$\rho > 0, \quad \|j\| \le \rho, \tag{11}$$

the system (2) is hyperbolic i.e. the matrix of transpport coefficient is diagonalizable (see [27] for detailled definitions). Second, the solution

of the Riemann problem without source terms lies in the set of admissible state. Assuming that a Godunov type numerical scheme with a splitting of the source terms converges, the solution will remain in the admissible set is invariant.

From the physical point of view, the admissible state comes from the fact that ρ represents a number of photons which has to be non negative and that the mean velocity of the photons j/ρ is smaller than the speed of light (which is normalized to 1 in the choosen scaling).

Note that the proof of convergence of the scheme is behind the scope of this paper. We shall verify on the numerical results in the last section that this convergence is expected.

Let us rewrite the system in variable $U = (\rho, u)$ with $j = \rho u$. The admissible states are characterized by $\rho \geq 0$ and $\|u\| \leq 1$. The matrix of transport coefficient $A(U)$ such that system (18) reads

$$\partial_t U + A(U)\partial_x U = R(U) \ , x \in \mathbb{R} \ , t > 0 \tag{12}$$

is given by

$$A(U) = \begin{pmatrix} u & \rho \\ \frac{h(u)-u^2}{\rho} & h'(u) - u \end{pmatrix} , \tag{13}$$

Its eigenvalues $\lambda_{\pm}$ are given by

$$\lambda_{\pm} = \frac{1}{2}\left(h'(u) \pm \sqrt{(h' - 2u)^2 + 4(h - u^2)}\right) \tag{14}$$

which are both real using hypothesis (10) on the h function .

The invariant property of the approximated solution is based on a splitting argument between the transport part and the source terms. We claim that both of the two operators preserve the properties (11), then the composition of the two will have the same property. It remains to prove that the splitting procedure converges as the time step goes to zero using a argument related to Trotter formula

$$\exp(A + B)t = \lim_{n\to\infty} (\exp(At/n). \exp(Bt/n))^n$$

The invariant property is obvious on the source terms. Indeed, ρ remains constant whereas $\|j\|$ decreases. For the transport part i.e. the system (2) with $\sigma = 0$, it can be checked that the properties hold true for the Riemann problem i.e. considering two states U_l and U_r, we verify that the 1-wave curve that contains U_l (and which consists of half shock curve and half rarefaction wave) intersects the 2-wave

curve that contains U_r in a so called intermediate state that is in the invariant domain.

More precisely, it can be proved that the 1-wave has the following behaviour in the ρ, u plane : the curve can be parametrized by u and the parametrization $\rho(u)$ is decreasing and satisfaies $\rho(u) \to +\infty$ as $u \to -1$, $\rho(u) \leq 0, \forall u \in]-1,1[$ and by construction $\rho(u_l) = u_l$. Similarly, the 2-wave can also be parametrized by u and is increasing, positive and $\lim_{u\to 1} \rho(u) = +\infty$, $\rho(u_r) = \rho_r$. Using the above properties of the 1 and 2-wave curves, one obtain the existence of an intermediate state (ρ_*, u_*) which satisfy the required properties (11). We refer to [3] for the details on the construction of the solution for the Riemann problem of system (2).

Then, if the Godunov type scheme converges, the invariant property will be satisfaied for the continuous solution of the transport part.

Remark 3 This invariant property (11) has a physical interpretation, since ρ and j the two first moments of the distribution function on the unit sphere $\rho = \int f d\omega, \quad j = \int f\omega d\omega$). From numerical point of view, this property means that the flux are so-called limited.

2.3 Asymptotic limit - diffusive regimes

Let us formally present the asymptotic limit of the system in so called diffusives regimes. In such scaling, the system (18) can be written in the form

$$\varepsilon \partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \tag{15}$$

where $U = (\rho, j)$, $F(U) = (j, \rho h(j/\rho)$, $R(U) = (0, -\sigma j)$, $\sigma(x) > 0$, the cross section and ε is a small parameter.

In the limit $\varepsilon \to 0$, a formal asymptotic gives that j is $O(\varepsilon)$ due to the collision term. At first order in ε, using the second equation of (15), we get

$$j = -\frac{\varepsilon}{\sigma} \partial_x (h(0)\rho). \tag{16}$$

This corresponds to suppress the time derivative term into the equation on j. Then, using $h(0) = 1/3$ and using (16) into the first equation, we obtain the following diffusion approximation

$$\frac{\partial}{\partial t}\rho - \frac{\partial}{\partial x}\left(\frac{1}{3\sigma}\frac{\partial}{\partial x}\rho\right) = 0. \tag{17}$$

Note that the solution for ρ of the limit heat equation (17) and j given by by (16) does not satisfy automatically the condition (11), because the gradient $\partial_x \rho$ can be arbitrarily large e.g. if the initial data is discontinuous in ρ. Note also that j or $\partial_x \rho$ are also solution of the same heat equation.

Our aim is now to design an implicit scheme compatible with the limit $\varepsilon \to 0$ and with the invariant property (11).

Various methods have been proposed to get ride of these difficulties such as variable Eddington factors for the so-called $P1$-approximation or flux limiters for diffusion approximation (see [18] and ref. therein). This is also related to a series of papers about asymptotic preserving schemes for kinetic problems, well balanced schemes, stiff source terms and relaxation methods in the context of hyperbolic systems [12, 8].

3 Numerical method for radiative model

Our method is based on a time splitting in two steps. The first step is based on a relaxation method [12] and the second on a well-balanced schemes [8].

Our goal is to solve the nonlinear problem while preserving positivity. Linearizing the equation is not suitable since the invariance domain will not be preserved and it will give wrong results when using implicit method.

3.1 The relaxed system.

Let us briefly recall the relaxation method according to [12]. Considering a system of the form

$$\partial_t u + \partial_x f(u) = 0,$$

it becomes in the relaxation limit $\alpha \to 0$

$$\partial_t (u, v) + \partial_x (v, au) = (0, -(f(u) - v)/\alpha).$$

This can be written in the form

$$\partial_t U + A \partial_x U = R(U), \tag{18}$$

where A is a constant matrix of transport coefficient and $R(U)$ is a , possibly nonlinear, source term.

Let us now apply this method to our system. The first step is to rewrite system (15) as the limit ($\alpha \to 0$) of the following relaxation system

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\partial_x z = 0 \\ \partial_t z + \frac{a}{\varepsilon}\partial_x \rho + \frac{\sigma}{\varepsilon^2} z = -\frac{1}{\alpha}(j - z) \\ \partial_t w + \frac{a}{\varepsilon}\partial_x j = \frac{1}{\alpha}(\rho h(\frac{j}{\rho}) - w) \\ \partial_t j + \frac{1}{\varepsilon}\partial_x w + \frac{\sigma}{\varepsilon^2} j = 0. \end{cases} \tag{19}$$

Note that, formally, the relaxed system is equivalent to (15) as the limit ($\alpha \to 0$) since the equilibrium states are given by

$$z = j, \quad w = \rho h(\frac{j}{\rho}). \tag{20}$$

At this point, the coefficient a remains to be choosen.

Our method consists in splitting the transport part or left hand side of system (19) and the relaxation term i.e. the right hand side. In the relaxation part, the original variables (ρ and j) are unchanged whereas the new variables (z and w) converge to the equilibrium state given by (20) in the limit $\alpha \to 0$. Thus, the relaxation part reduces into a projection on equilibrium states. The coefficient a is constant in space but has to be choosen at each time step in order to recover the correct diffusion coefficient.

The transport part writes:

$$\begin{cases} \partial_t \rho + \frac{1}{\varepsilon}\partial_x z = 0 \\ \partial_t z + \frac{a}{\varepsilon}\partial_x \rho + \frac{\sigma}{\varepsilon^2} z = 0 \\ \partial_t w + \frac{a}{\varepsilon}\partial_x j = 0 \\ \partial_t j + \frac{1}{\varepsilon}\partial_x w + \frac{\sigma}{\varepsilon^2} j = 0. \end{cases} \tag{21}$$

Let us point out that the relaxation term on the equation over z is not classical. However, this term is very important in order to get the right asymptotic behaviour in diffusives regimes i.e. when $\varepsilon \to 0$.

Let us also emphasize that (21) is just two linear and identical systems. These systems are uncoupled : one for the quantities (ρ and z), the second for (w and a new variable $\bar{j} = aj$) which are equivalent to the system (15) with $h \equiv a$.

Note that this system i.e. (15) with $h \equiv a$ once diagonalized is a well-know Goldstein-Taylor or Telegraph equation with speed $\pm\sqrt{a}$

i.e. two transport equations in opposite direction coupled by a relaxation term

$$\begin{cases} \partial_t u + \frac{\sqrt{a}}{\varepsilon}\partial_x u = \frac{\sigma}{2\varepsilon^2}(v-u) \\ \partial_t v - \frac{\sqrt{a}}{\varepsilon}\partial_x v = \frac{\sigma}{2\varepsilon^2}(u-v). \end{cases} \tag{22}$$

Let us mention that the invariant domain property (11) can be seen as the positivity of the transported quantities $\rho \pm j > 0$. In the above system, the transport quantities (u or v) are $\rho \pm z/\sqrt{a}$ (or $j \pm w/\sqrt{a}$). We shall choose the coefficient a in such a way that the transport quantities remain positive.

Using the following change of variable

$$\begin{aligned} U &= \sqrt{a}\rho + z + w + \sqrt{a}j, \\ V &= \sqrt{a}\rho - z + w - \sqrt{a}j, \\ \bar{U} &= \sqrt{a}\rho + z - w - \sqrt{a}j, \\ \bar{V} &= \sqrt{a}\rho - z - w + \sqrt{a}j, \end{aligned} \tag{23}$$

we verify that (U,V) and $(\bar{U},\bar{V})$ satisfy a system of the form (22). We will show that for the transport part, the invariant domain (11) comes from the positivity of $U, V, \bar{U}, \bar{V}$ for sufficiently large value of a.

The initial data for the new variables verify, using the equality of the projected variable at initial time $z = j$ and $w = \rho h(u)$, become

$$\begin{aligned} U &= \rho(\sqrt{a} + h(u)) + j(\sqrt{a}+1), \\ V &= \rho(\sqrt{a} + h(u)) - j(\sqrt{a}+1), \\ \bar{U} &= \rho(\sqrt{a} - h(u)) - j(\sqrt{a}-1), \\ \bar{V} &= \rho(\sqrt{a} + h(u)) + j(\sqrt{a}-1). \end{aligned} \tag{24}$$

Then, using that $j = \rho u$ and $\rho \geq 0$, we obtain that U at $t = 0$ is positive provided that

$$\sqrt{a} + h(u) + (\sqrt{a}+1)u \geq 0.$$

Let us assume that u (at initial time) lies in the interval $[-b, b]$ with some $b < 1$. We have to choose a. Note that , for any $a > b$ and any $u \in [-b, b]$,

$$\sqrt{a} + h(u) + (\sqrt{a}+1)u \geq \sqrt{a} + h(b) - (\sqrt{a}+1)b.$$

Thus, the following value

$$\sqrt{a} = \frac{h(b) - b}{1 - b},$$

insurses positivity. Any larger value of a will also be convenient, for example

$$a \stackrel{def}{=} h(b). \tag{25}$$

Indeed, it can be easily verify that this choice preserves positivity :

$$\sqrt{a}+h(u)+(\sqrt{a}+1)u \geq \sqrt{h(b)}+h(b)-(\sqrt{h(b)}+1)b = (\sqrt{h(b)}+1)(\sqrt{h(b)}-b) \geq 0,$$

since $h(y) \geq y^2$. The properties hold also the the other three quantities $V, \bar{U}, \bar{V}$.

Thus, we prove that the following choice for the coefficient a

$$a = h(\max_{x\in\mathbb{R}} \|u(x)\|),$$

ensurses the positivity of the initial data for $U, V, \bar{U}, \bar{V}$ and then of the solution $U, V, \bar{U}, \bar{V}$ of the system (21). The sketch of the proof for the positivity is similar to the one of section 2.2. We consider the two independant systems of the form (22). We construct the solution of such system as the limit of a approximated solution based on a splitting. For the transport part (as speed $\pm\sqrt{a}$) the positivity of both u and v is obvious. For the source terms, we easily check that the solution of the relaxation

$$\partial_t u = \frac{\sigma}{2\varepsilon^2}(v-u), \quad \partial_t v = \frac{\sigma}{2\varepsilon^2}(v-u),$$

also preserves positivity. Thus, the approximation satisfy the properties and so does its limit. We assume the convergence of such splitting based algorithms, which is reasonnable for any fixed ε.

In this case, we can prove directly the above result : For any solution of (22) with positive initial data, the solution remains positive. We define the postive and negative part of a fonction f and denote it by f^+ and f^-

$$f^+ = (f + \mid f \mid)/2, f^- = (f - \mid f \mid)/2.$$

We multiply the equations of (22) by u^- and v^- respectively and we integrate for $x \in \mathbb{R}$

$$\begin{cases} \partial_t \int u^- u dx + \frac{\sqrt{a}}{\varepsilon} \int u^- \partial_x u = \int \frac{\sigma}{2\varepsilon^2} u^-(v-u) \\ \partial_t \int v^- v dx - \frac{\sqrt{a}}{\varepsilon} \int v^- \partial_x v = \int \frac{\sigma}{2\varepsilon^2} v^-(u-v). \end{cases}$$

We have $u^- u = (u^-)^2$ and the integral of $\partial_x (u^-)^2$ is zero assuming $u(x,t) \to 0$ when $x \to \pm\infty$. Thus,

$$\partial_t \int (u^-)^2 + (v^-)^2 dx = \int \frac{\sigma}{2\varepsilon^2}(u^-(v-u) + v^-(u-v)) \leq 0.$$

Indeed,

$$u^-(v-u)+v^-(u-v) = u^-(v^++v^--u^-)+v^-(u^++u^--v^-) = u^-v^++u^+v^--(u^--v^-)^2.$$

The first two terms are non positive by definition of (u^-, v^+) and (u^+, v^-) respectively.

The initial data being positive implies $u^-(t=0) = v^-(t=0) = 0$ and the above inequality proves that the L^2 norm of u^- and v^- decay. This proves that u^- and v^- vanish for any time i.e. that the solution u and v remain positive.

Remark 4 The proposed choice for a is not satisfactory since it depends on the whole solution $u(x)$ for $x \in \mathbb{R}$. We refer to [6] for a variant of the solution proposed here that overcomes this difficulty.

In the diffusive limit ($\varepsilon \to 0$) or for large time behaviour, we expect that $\max_{x\in\mathbb{R}} \|u(x)\| \to 0$ and therefore, a will become close to $1/3$ i.e. we obtain the right asymptotic (17).

Remark 5 Despite its linear structure, the system (22) give raise to severe numerical problems. It can also be seen as a very simple model of kinetic theory of gases where particles can only have velocity $\pm\sqrt{a}$. In this context, the limit $\varepsilon \to 0$ corresponds to a diffusion limit which a diffusion coefficient equal to $\frac{a}{\sigma}$.

3.2 An interpretation of the Well Balanced scheme

We shall now solve numerically the system (22) for (ρ, z) (and similarly for the other identical system in variable (w, aj)).
We introduce a non-uniform mesh : we note x_i, the center of the cell of size Δx_i with $i \in \mathbb{Z}$ and define $\Delta x_{i+\frac{1}{2}} = (\Delta x_i + \Delta x_{i+1})/2$.

In this part, we shall present an interpretation of the WB scheme for the so called telegraph equation and/or Goldstein-Taylor model

$$\partial_t u + \partial_x u = -\sigma(v-u), \partial_t v - \partial_x v = \sigma(v-u). \tag{26}$$

This system, with source term , can be rewritten in a more compact form

$$\partial_t U + A\partial_x U = R(U).$$

One possible way to recover the WB scheme is to approximate the source term using a quadrature formulae that localize the source at interface i.e. $R(U)$ is replaced by

$$\sum_i \delta(x - x_{i+\frac{1}{2}})\Delta x_{i+\frac{1}{2}} R(U_{i+\frac{1}{2}}).$$

Let us now introduce an extended non conservative hyperbolic system (without source term) for (U, id) where id represents the identity function (constant in time)

$$\begin{cases} \partial_t U + A\partial_x U = R(U)\partial_x id, \\ \partial_t id = 0, \end{cases} \tag{27}$$

with the matrix

$$B = \begin{pmatrix} A & -R \\ 0 & 0 \end{pmatrix}.$$

Note that since A is diagonlizable, B too and its spectrum consists of the eigenvalues of A and zero. Let us now assume that the matrix A is diagonal. Then, using a piecewise constant approximation of the auxillary function $id(x)$ (and thus its derivative becomes sum of delta function localized at interfaces $x_{i+\frac{1}{2}}$ with weights $\Delta x_{i+\frac{1}{2}}$) yields to the quadrature formulae proposed above.

This way of introducing the localization of the source at interfaces permits to extend naturally this approach to non uniform mesh and to multi-dimensional problem.

We have now to solve the Riemann problem including the source term. Thus, we localize the analysis near the interface, assuming the time step small enough such that the waves will not interact from one cell to another. We have assumed A is diagonal, thus we can treat each component separately i.e. consider that U is a scalar and the matrix A reduces to a real number. Let us assume $A > 0$ for example.

Let us now introduce a mollifier sequence χ_β of the Dirac measure, for example, we can choose characteristic function with vanishing support $\chi_\beta(x) = \frac{1}{\beta}$ for any $\|x\| < \beta/2$ and 0 elsewhere. The regularized local Riemann problem with source reads for $\|x\| < \beta/2$

$$\partial_t U_\beta + A\partial_x U_\beta = \frac{1}{\beta} R(U_\beta)\Delta x_{i+\frac{1}{2}},$$

and the initial data $U(x, t = 0) = U_L$ for $x < 0$ and $U(x, t = 0) = U_R$ for $x > 0$. Since $a > 0$ the transport propagates to the right, the solution of this Riemann problem satisfaies

$$\begin{cases} U(x,t) = U_L, \ \ \forall (x,t) s.t. x < -\beta/2, \\ U(x,t) = U_R, \ \ \forall (x,t) s.t. x > At + \beta/2, \\ U(x,t) = U_*, \ \ \forall (x,t) s.t. \beta/2x < At + \beta/2, \end{cases}$$

where U_* is the outgoing value of U associated to the entering value U_L after crossing the interval $[-\beta/2, \beta/2]$. Note that we do not detail the solution inside the interval since this interval is vanishing in the limit $\beta \to 0$, but we need to compute the outgoing value U_*. When rescaling the interval $y = x/\beta$, the problem becomes

$$\beta\partial_t U(y,t) + A\partial_y U(y,t) = R(U)\Delta x_{i+\frac{1}{2}},$$

and formally, when $\beta \to 0$ the problem becomes stationnary. Thus, in the limit $\beta \to 0$ the outgoing value U_* is given by the stationnary solution.

Let us now solve the stationary problem for the telegraph equation of interest and an arbitrary mollifier. Setting $\rho = (u+v)/2$ and $j = (u-v)/2$, the stationnary equation (in variable y) associated with (26) reads (let us assume $\sigma = 1$ for simplifying the notations).

$$\partial_y \rho = -\chi(y) j, \quad \partial_y j = 0.$$

The current j is constant (equal to $u_L - v_L$ and to $u_R - v_R$) and the equation for ρ gives

$$\rho_R - \rho_L = j \int_{-1/2}^{1/2} \chi(y) dy = j.$$

Thus, the stationnary solution is characterized by the equations

$$\rho_R - \rho_L = j = j_R = j_L.$$

These equations are independent of the choosen mollifier.

For the telegraph equation ($A = 1$), the eigenvalues are ± 1 and the Riemann invariant are the functions u and v respectively. Then, the solution of the Riemann problem with a localized source terms is defined by

$$\begin{cases} U(x,t) = U_L = (u_L, v_L), & \forall (x,t) s.t. x < -t, \\ U(x,t) = U_L^* = (u_L, v_L^*), & \forall (x,t) s.t. -t \geq x < 0, \\ U(x,t) = U_R^* = (u_R^*, v_R), & \forall (x,t) s.t. 0 \geq x < t, \\ U(x,t) = U_R = (u_R, v_R), & \forall (x,t) s.t. x \geq t, \end{cases}$$

Indeed, the Riemann invariant u_L is constant through the left wave with speed -1. The intermediate states U_L^* and U_R^* are connected by a

stationnary wave and thus satisfy the above relations. Once expressed in the original variable ρ and j, the solution is uniquely defined by

$$\begin{cases} j_R^* = j_L^* = j_0, \quad \rho_R^* - \rho_L^* = -j_0 \Delta x, \\ \rho_L^* + j_0 = u_L, \\ \rho_R^* - j_0 = v_D. \end{cases}$$

The first equations are nothing but the relations for states connected by a stationnary wave. The last two equations come from the conservation of the u (resp. v) through the wave of speed -1 (resp. $+1$). The solution of this linear system is

$$j_0 = \frac{u_L - u_R}{2 + \Delta x}, \quad \rho_L^* = u_L - j_0, \quad \rho_R^* = v_D + j_0.$$

Then, one can write a Godunov scheme using these exact solution for the Riemann problem.

When performing the same analysis i.e. solving the Riemann problem with a cross section σ/ε instead of σ), we find

$$j_0 = \frac{u_L - u_R}{2 + \frac{\sigma \Delta x}{\varepsilon}},$$

and with velocity $\pm\sqrt{a}$ (as in (22) instead of ± 1 (as in (26), we have to replace Δx by $\Delta x/\sqrt{a}$ and, thus, the coefficient becomes

$$j_0 = M(u_L - u_R), \quad M = \frac{2\varepsilon\sqrt{a}}{2\varepsilon\sqrt{a} + \sigma \Delta x}.$$

Last, we have to project onto piecewise constant solution at iteration $n+1$. The average value over the cell is given by

$$U_{i+\frac{1}{2}}^{n+1} = \frac{A \Delta t U_* + (\Delta x_{i+\frac{1}{2}} - A \Delta t) U_{i+1}}{\Delta x_{i+\frac{1}{2}}}.$$

We have a similar formula with U_i instead of U_{i+1} in the case $A < 0$.

Let us now integrate - in time and space - the obtained solution onto the cell without the thin boundary layer near interfaces i.e. we integrate the equation onto $[x_{i-\frac{1}{2}} + \varepsilon/2, x_{i+\frac{1}{2}} - \varepsilon/2] \times [t^n, t^{n+1}[$. By construction, the time derivative gives the difference $U_i^{n+1} - U_i^n$ and the source term is identically zero. We obtain

$$U_i^{n+1} - U_i^n + \frac{1}{\Delta x_i} \int_0^{\Delta t} F(U_{i-\frac{1}{2}}^{\varepsilon}(s)) - F(U_{i-\frac{1}{2}}^{\varepsilon}(s)) ds = 0.$$

The obtained scheme is a direct extension of the, so called well balanced scheme described in [8] for telegraph equation with velocity $\pm\sqrt{a}$, variable cross section σ and non uniform mesh:

$$\begin{cases} \frac{\Delta u_i}{\Delta t} + M_{i-\frac{1}{2}} \frac{\sqrt{a}}{\varepsilon \Delta x_i}(u_i - u_{i-1}) = M_{i-\frac{1}{2}} \frac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i} \frac{\sigma_{i-\frac{1}{2}}}{2\varepsilon^2}(v_i - u_i) \\ \frac{\Delta v_i}{\Delta t} - M_{i+\frac{1}{2}} \frac{\sqrt{a}}{\varepsilon \Delta x_i}(v_{i+1} - v_i) = M_{i+\frac{1}{2}} \frac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i} \frac{\sigma_{i+\frac{1}{2}}}{2\varepsilon^2}(u_i - v_i). \end{cases} \tag{28}$$

where $\frac{\Delta u_i}{\Delta t}$ denotes either the partial derivative of u_i with respect to t i.e. a semi-discretized system or a time discretization (e.g $\frac{u_i^{n+1} - u^n}{\Delta t}$) and the coefficient $M_{i+\frac{1}{2}}$ defined by

$$M_{i+\frac{1}{2}} = \frac{2\sqrt{a}\varepsilon}{\sigma_{i+\frac{1}{2}} \Delta x_{i+\frac{1}{2}} + 2\sqrt{a}\varepsilon}, \tag{29}$$

with $\sqrt{a}$ is the constant value of the limit diffusion coefficient as defined above and $\sigma_{i+\frac{1}{2}}$ is an arbitrary average of σ at interface (e.g. arithmetic, harmonic...). The above scheme corresponds to the one proposed in [8] for a uniform mesh, $\sigma = 2$ and a diffusion coefficient in the limit heat equation equal to $\frac{1}{2}$. Note that, in our case, the cross section is not assumed to be constant, which is of main interest from applications point of view.

We can show that (28) is a monotone scheme and then (11) remains an invariant domain during the transport part. It is readily seen that, in the limit $\max_i(\Delta x_i) \to 0$, the coefficient $M_{i+\frac{1}{2}}$ tends to 1 and the consistency of the scheme (28) with the continuous system (22) is satisfied provided that, in the limit, the mesh is smooth enough i.e. that is locally an uniform mesh $(\frac{\Delta x_{i+1}}{\Delta x_i}) \to 1$ when $\max_i(\Delta x_i) \to 0$.

Note that the proposed scheme (28) can be simplified for an uniform mesh and constant cross section. In this case, the coefficient M is also constant and the equation for $\frac{\Delta u_i}{\Delta t}$ reads

$$\frac{\Delta u_i}{\Delta t} + M \frac{\sqrt{a}}{\varepsilon \Delta x}(u_i - u_{i-1}) = M \frac{\sigma}{2\varepsilon^2}(v_i - u_i)$$

can be equivalently writen as

$$\frac{\Delta u_i}{\Delta t} + \frac{\sqrt{a}}{\Delta x \varepsilon} u_i = M(\frac{\sigma}{2\varepsilon^2} v_i + \frac{\sqrt{a}}{\varepsilon \Delta x} u_{i-1}). \tag{30}$$

This form is suitable for insuring the positivity of the solution or, equivalently, the condition $\|j\| \le \rho$) i.e. $u_i(t = 0) > 0 \forall i \in \mathbb{Z} \quad \Rightarrow$

$u_i(t) > 0,\ \forall i,\ \forall t > 0$ either for the semi-discretized system (when $\frac{\Delta u_i}{\Delta t}$ represents $\frac{\partial u_i}{\partial t}$) or for semi-implicit schemes (when it represents $\frac{u_i^{n+1}-u^n}{\Delta t}$). For semi-discretized system we have

$$u_i(t) \geq u_i(0)\exp(-\frac{\sqrt{a}}{\Delta x\varepsilon}t).$$

A similar argument holds for the positivity of v.

Let us mention a last equivalent forms of the scheme with uncentred source terms like in [1]

$$\frac{\Delta u_i}{\Delta t} + \frac{\sqrt{a}}{\Delta x\varepsilon}(u_i - v_i) = M\frac{\sqrt{a}}{\Delta x\varepsilon}(u_{i-1} - v_i). \tag{31}$$

These equivalences come from the equality

$$M(\frac{\sqrt{a}}{\Delta x\varepsilon} + \frac{\sigma}{2\varepsilon^2}) = \frac{\sqrt{a}}{\Delta x\varepsilon}.$$

3.3 Interpretation as HLL scheme

Let us now interpret the obtained scheme in terms of the so called Harten-Lax-Van Leer scheme described in [27].

Indeed, the scheme (28) can be writen in the original variables (ρ, z) and the same for $(w, \bar{j})$

$$\begin{cases} \frac{\partial \rho_i}{\partial t} + \frac{1}{\varepsilon\Delta x_i}(M_{i+\frac{1}{2}}z_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}z_{i-\frac{1}{2}}) = 0, \\ \frac{\partial z_i}{\partial t} + \frac{a}{\varepsilon\Delta x_i}(M_{i+\frac{1}{2}}\rho_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}\rho_{i-\frac{1}{2}}) = \frac{-\lambda_i}{2\varepsilon^2}z_i + \frac{M_{i+\frac{1}{2}} - M_{i-\frac{1}{2}}}{\varepsilon\Delta x_i}(a\rho_i), \end{cases} \tag{32}$$

with

$$z_{i+\frac{1}{2}} = (z_i + z_{i+1} + \rho_{i+1} - \rho_i)/2, \quad \rho_{i+\frac{1}{2}} = (\rho_i + \rho_{i+1} + z_{i+1} - z_i)/2,$$

and

$$\lambda_i = \frac{\Delta x_{i+\frac{1}{2}}}{\Delta x_i}M_{i+\frac{1}{2}}\sigma_{i+\frac{1}{2}} + \frac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i}M_{i-\frac{1}{2}}\sigma_{i-\frac{1}{2}}.$$

Once again, the consistency of the scheme (when $\Delta x \to 0$ for fixed ε) requires a asymptotically regular mesh . More precisely, this means that, when the mesh is refined, it becomes locally regular $\frac{\Delta x_{i-\frac{1}{2}}}{\Delta x_i} \to 1$ as $\max_i \Delta x_i \to 0$.

One can also check the asymptotics $\varepsilon \to 0$ (with fixed Δx_i presumably non uniform) using the above form. It is readily seen that z

has to remain small and, more precisely, of the order of magnitude of $O(\varepsilon)$. On the other hand, the limit behaviour of M is given by

$$M_{i+\frac{1}{2}} \sim \frac{2\varepsilon\sqrt{a}}{\sigma_{i+\frac{1}{2}}\Delta x_{i+\frac{1}{2}}},$$

and for the λ coefficient

$$\lambda_i \sim \frac{4\varepsilon\sqrt{a}}{\Delta x_i}.$$

Moreover, the last term of the r.h.s. is small and the leading order of the equation for z is

$$\frac{a}{\varepsilon\Delta x_i}\left(\frac{2\varepsilon}{\sigma_{i+\frac{1}{2}}\Delta x_{i+\frac{1}{2}}}\rho_{i+\frac{1}{2}} - \frac{2\varepsilon}{\sigma_{i-\frac{1}{2}}\Delta x_{i-\frac{1}{2}}}\rho_{i-\frac{1}{2}}\right) = \frac{-4\varepsilon}{2\sqrt{a}\varepsilon^2\Delta x_i}z_i.$$

Furthermore, in the limit $\varepsilon \to 0$, we have (since z_i is of order ε)

$$z_{i+\frac{1}{2}} = (\rho_{i+1} - \rho_i)/2.$$

Then, reporting the last expressions in the equation for ρ_i, one obtain the discretization of the heat equation on a nonuniform grid. Indeed, retaining only the first order terms in the preceeding expressions, the equation for ρ_i in (32) becomes

$$\frac{\partial\rho_i}{\partial t} + \frac{\sqrt{a}}{\Delta x_i}\left(\frac{\rho_{i+1}-\rho_i}{\Delta x_{i+\frac{1}{2}}} - \frac{\rho_i - \rho_{i-1}}{\Delta x_{i-\frac{1}{2}}}\right) = 0.$$

Note that the formulae can be simplified for uniform mesh and constant cross section : in this case, we have $\lambda = 2\sigma M$ and the second term of the right hand side vanishes. Then, the proposed scheme reduces to a classical Godunov scheme

$$\varepsilon\frac{\partial U_i}{\partial t} + M(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}})/\Delta x = \frac{M}{\varepsilon}R(U_i), \tag{33}$$

where the flux at interfaces are given by

$$F_{i+\frac{1}{2}} = [b_i F_{i+1} + b_{i+1}F_i + b_i b_{i+1}(U_{i+1} - U_i)]/(b_i + b_{i+1}), \tag{34}$$

which are just upwind fluxes for(22) with $b_i = \sqrt{a}$ The multiplicative coefficient $M = \frac{2D\varepsilon}{\sigma\Delta x + 2D\varepsilon}$ comes from the well balanced scheme where D is the expected diffusion coefficient i.e. $h(0)$.

This form is useful because it is expressed in the original variable and this can be easily generalized to nonlinear case. The discretization is a sum of a diffusive term and classical convective term. Thus, this discretization can be interpreted as a particular choice of adding numerical viscosity depending of ε.

3.4 Comparison

Let us now detail the comparison with other proposed scheme for similar models.

For example, the scheme proposed in [15] can be written, using our notations, as an interpolated scheme between an upwing scheme and a centrered one. Starting from the system in the form (2) with $h = 1$ i.e. the telegraph equation, we set $j = \varepsilon \bar{j}$

$$\begin{cases} \partial_t \rho + \partial_x \bar{j} &= 0 \\ \partial_t \bar{j} + (1 + (1-\varepsilon^2)/\varepsilon^2)\partial_x \rho &= \frac{-\sigma}{\varepsilon^2}\bar{j}. \end{cases} \tag{35}$$

Let us split the system into a transport part with velocity ± 1 which is discretized using a upwind scheme (in variable u and v) and the remaining part of the term $\partial_x(\rho h)$ is taken as a source term, using a centrered scheme. Once back into the original variable $(\rho, \bar{j})$, the semi-discretized system, on a uniform grid in space, reads

$$\begin{cases} \partial_t \rho_i + \frac{1}{2\Delta x}(\bar{j}_{i+1} - \bar{j}_{i-1} - (\rho_{i+1} + \rho_{i-1} - 2\rho_i)) = 0 \\ \partial_t \bar{j}_i + \frac{1}{2\Delta x}(\rho_{i+1} - \rho_{i-1} - (\bar{j}_{i+1} + \bar{j}_{i-1} - 2\bar{j}_i)) = \\ \qquad \frac{-1}{\varepsilon^2}\left[\sigma \bar{j}_i + (1-\varepsilon^2)(\rho_{i+1} - \rho_{i-1})/(2\Delta x)\right]. \end{cases} \tag{36}$$

The above discretization can be written equivalently in ρ, j as

$$\begin{cases} \partial_t \rho_i + \frac{1}{\varepsilon}\frac{1}{2\Delta x}(j_{i+1} - j_{i-1}) - \frac{1}{2\Delta x}(\rho_{i+1} + \rho_{i-1} - 2\rho_i) = 0 \\ \partial_t j_i + \frac{1}{\varepsilon}\frac{1}{2\Delta x}(\rho_{i+1} - \rho_{i-1}) - \frac{1}{2\Delta x}(j_{i+1} + j_{i-1} - 2j_i) = \frac{-\sigma}{\varepsilon^2} j_i \end{cases} \tag{37}$$

or,

$$\begin{cases} \partial_t \rho_i + \frac{1}{\varepsilon}\frac{1}{2\Delta x}\left[(1-\varepsilon)(j_{i+1} - j_{i-1}) + \right. \\ \qquad \left. +\varepsilon(j_{i+1} - j_{i-1} - (\rho_{i+1} + \rho_{i-1} - 2\rho_i))\right] = 0 \\ \partial_t j_i + \frac{1}{\varepsilon}\frac{1}{2\Delta x}\left[(1-\varepsilon)(\rho_{i+1} - \rho_{i-1}) + \right. \\ \qquad \left. +\varepsilon(\rho_{i+1} - \rho_{i-1} - (j_{i+1} + j_{i-1} - 2j_i))\right] = \frac{-\sigma}{\varepsilon^2} j_i \end{cases} \tag{38}$$

i.e. a centered scheme for the $(1-\varepsilon)$ part and the upwind scheme for the ε part.

The proposed scheme can also be related to scheme proposed recently in [1,20] and compared with previous works like [11,17,15].

4 Numerical results

We shall now present 2 numerical tests. The first is a validation of our method with a strongly variable cross section. The second case is a more complex case with a coupling with heat equation for material. The scheme is implicit in time which leads to the solving of a band matrix system.

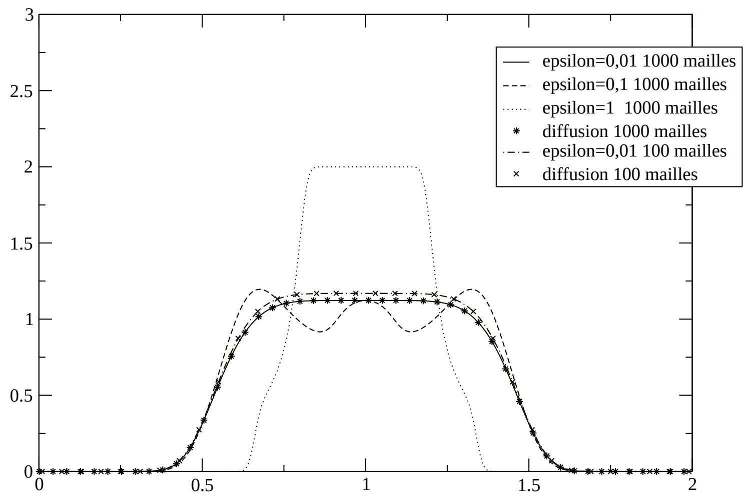
4.1 Variable opacity coefficient

Let us consider the case

$$x \in [0,2], \sigma(x) = 100(x-1)^4.$$

This corresponds to a case of transparent media in the center ($\sigma = 0$ for $x = 1$) with opaque walls at boundary ($x = 0$ and $x = 2$).

The initial data is a characteristic function, for ρ with support in $[\frac{1}{2}, \frac{3}{2}]$. The initial flux j is equal 0. The simulated time is $T = 0.1$ and the small parameter value takes the following values : $\varepsilon = 0$ (i.e. the diffusion case) and the following values $10^{-2}, 0.1, 1$. The mesh is uniform with either 100 or 1000 points and the time step is chosen such that $\Delta t/\Delta x = 0.05$. Note that the expected time step restriction for the transport part (C.F.L. condition) is much more restrictive $\frac{\Delta t}{\Delta x} \leq \varepsilon$, and, similarly, the characteristic relaxation time is such that $\Delta t\sigma \leq \varepsilon^2$.



The computation using various number of discretization points indicates that the solution converges when the mesh size (and thus the time step) goeas to 0. This is some kind of numerical consistency result. On the other hand, when ε goes to 0, we obtain a solution that converges toward the one of a diffusion equation with the diffusion coefficient 1/3. This illustrates that the proposed scheme is compatible with the diffusive asymptotics.

4.2 Coupling with material

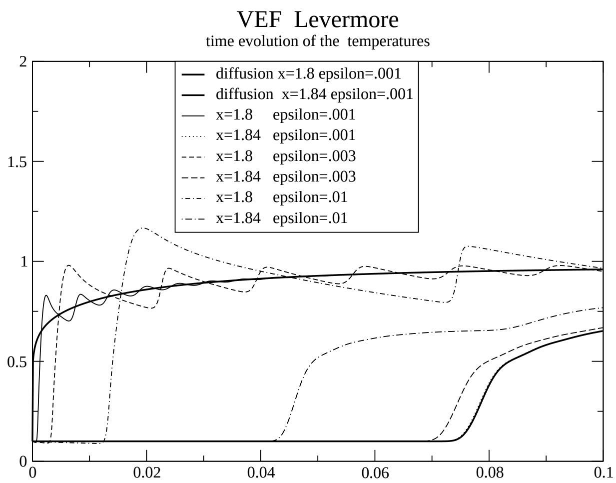
Let us consider the system coupled with an heat equation for some material. This test can be seen as a simplified model of laser-plasma interaction

$$\begin{cases} \varepsilon\partial_t\rho + \partial_x j = \frac{\tau}{\varepsilon}(K-\rho) \\ \varepsilon\partial_t j + \partial_x \rho h(j/\rho) = -\frac{\tau+\sigma}{\varepsilon} j \\ C_v\partial_t T = \frac{-\tau}{\varepsilon^2}(K-\rho), \end{cases} \tag{39}$$

σ is the scattering coefficient for photons, τ is the absorption coefficient of the material. K can be either given or equal to T^4 following Stefan law. We have $\tau = \frac{C_v}{T^3\varepsilon^2}$.

The choosen closure relation is the Levermore-Lorentz one [19,18] given by (7). We are using a time splitting :one time step for the moment systems in (ρ, j) with fixed temperature T and then, the system for (ρ, T). The first test consists of a domain, scaled to $[0, 2]$, between 2 walls. Let us summarize the scaled parameters of the simulation

	x<0.1	0.1 < x < 0.2	0.2 < x < 1.8	1.8 < x < 1.9	x > 1.9
σ	$+\infty$	0	0	0	$+\infty$
C	0	10	0	10	0
ρ^0	0	16	0	10^{-4}	0
j^0	0	0	0	0	0
T^0	0	2	0	10^{-1}	0



Initially, the left wall is hot and, then, by radiation, it warms the right one. We plot the evolution of temperature due to this heating

at the surface $x = 1.8$, and inside the right wall for $x = 1.84$.The computations were made with $\Delta x = 1/500$ and $\Delta t = 10^{-4}$.

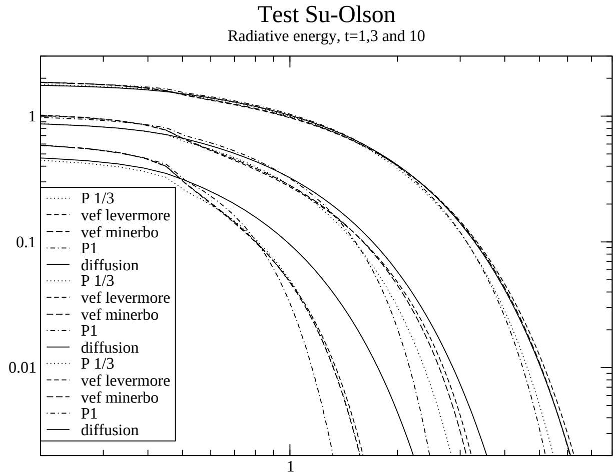
We observe on figure 2 that the solution converges toward those of the diffusion model when $\varepsilon \to 0$ as expected. Note that the dependance with respect to ε is weaker in the material (at $x = 1.84$) than at the surface (at $x = 1.8$). The front of heating reaches the surface $x = 1.8$ at time $t = 0.012$ for $\varepsilon = 0.01$ and this time goes to zero as $\varepsilon \to 0$ as expected for a diffusive equation. The oscillations (in time) of the temperature at $x = 1.8$ (surface of the material) are due to the heat wave that rebounds between the walls. The speed of the wave goes to infitnity as $\varepsilon \to 0$ and in the diffusive limit, the heat propagates instantaneously and, thus, the oscillations disappear. The front of heat is less sharp within the material and the delay to warm the material increases as $\varepsilon \to 0$ (from $t = 0.05$ at $\varepsilon = 0.01$ to $t = 0.075$ when $\varepsilon \to 0$).

The second test is taken from [21], page 625, with a constant opacity but with $C_v = \alpha T^3$ and a source term $S = 1$ localized in $\|x\| \leq 1/2$. All coefficients in (39) are taken to one thus the system of equations to solve is

$$\begin{cases} \partial_t \rho + \partial_x j = (K - \rho) + S \\ \partial_t j + \partial_x \rho h(j/\rho) = -j \\ \partial_t K = -(K - \rho). \end{cases} \tag{40}$$

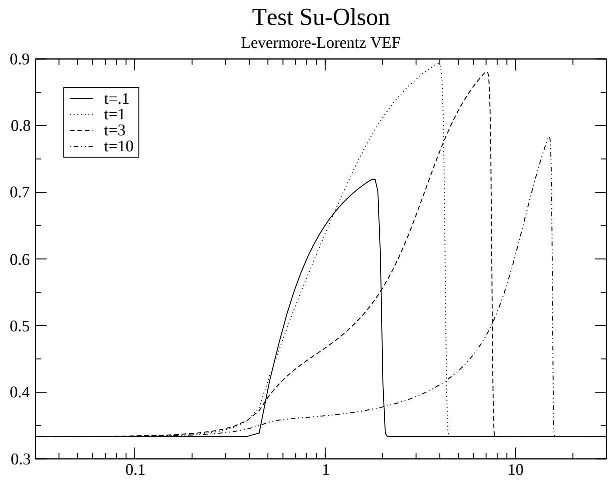
The computations were made with $\Delta x = 3/50$ and $\Delta t = 1/100$. Solutions, with different Eddington factors are viewed at times $t = .1, 1, 3$ and $t = 10$.

At early time, the discrepancies between various variable Eddington factors (VEF) and diffusion are important since diffusion is a less accurate model for short time. These differences, as expected , tend to reduce to zero as the time growing. At time $t = 10$, these discrepancies are negligible. We also show the results for the $P1$ model with$h = 1/3$ and for the, so called, $P_{1/3}$ model derived from the $P1$, (see [21] for a detailled presentation), by taken $h = 1$ and by multiplying the opacity by a factor 3. For these two constant Eddington factors, the discrepancies with diffusion results, still exist at time $t = 10$. This proves that is is important to use the correct closure relation or Eddington factor in order to recover the right behaviour of the solutions in particular at early time but also after rather long time.



Moreover, our results are very close to those obtained in [21], with a important difference : our scheme for variable Eddington factors gives no noisy solutions, in contrast with the corresponding results presented in [21]. This fact illustrates the robustness of our scheme.

We also show the profile of the Levermore-Lorentz Eddington factor at the same times.



for which we can give the same conclusion as for the precedent figure: no noisy solution for variables Eddington factors with our scheme. Thus, the instability or noise observed for the VEF calculations in [21] are certainly due to a bad discretization of nonlinear hyperbolic problem.

5 Conclusions

The proposed scheme has all the required properties announced in the introduction.

The scheme consists of two steps : first to replace the nonlinear into two independant and identical linear system of telegraph equations and, second, the use of a well balanced scheme for each of the two systems. The interpretation of the well balanced scheme as a Godunov scheme using the Riemann solution of hyperbolic system with source term (section 3.2) can be extended to more complex relaxation term.

A first example arises from relativistic effect as presented in the models described in [4]. The obtained equation is, in such case, a Burgers equation with diffusion instead of the heat equation in the case considered here. It can also be extended to discrete velocity models of kinetic equation in the diffusive regime with a linear collision operator of Lorentz type (see [2] for a detailled presentation). A third extension is to consider two dimensional (in space) model. The use of alternate direction (i.e. a splitting between the x and y direction) is, in diffusive regime, not suitable. Our method yields to choose distincts quadrature points for the transport in direction x and y respectively, or, in other words, it yields to localize the source terms at interfaces.

Moreover, the proposed approach can be combined with adaptative mesh refinement technics since it requires to evaluate the flux at interface by solving stationnary Riemann problem as explained in subsections 3.2 and 3.3. These are, in our opinion, promissing direction for forthcoming developpements of the proposed scheme.

The main drawback of the proposed scheme is that the choice of the velocity a for the relaxed system (2) is non local and thus, the diffusive regime is obtain only when the whole domain is in isotropic equilibrium. This is a rather severe limitation for the use of this scheme in complex situations. One possible solution is to use domain decompo-

sition and to use a different value of a in the different domains. We also mention a forthcoming paper [6] that proposed a variant of the proposed scheme for which the choice of the coefficient a is no more local.

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